Ensemble Kalman filtering with residual nudging
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Summary
Covariance inflation and localization are two important techniques that are used to improve the performance of the ensemble Kalman filter (EnKF) by adjusting the sample covariances of the estimates in the state space. In this work an additional auxiliary technique, called residual nudging, is proposed to monitor and, if necessary, adjust the residual norms of state estimates in the observation space. In an EnKF with residual nudging, if the residual norm of an analysis is larger than a pre-specified value, then the analysis is replaced by a new one whose residual norm is no larger than a pre-specified value. Otherwise the analysis is considered as a reasonable estimate and no change is made. A rule for choosing the pre-specified value is suggested. Based on this rule, the corresponding new state estimates are explicitly derived in case of linear observations. Numerical experiments in the 40-dimensional Lorenz 96 model show that introducing residual nudging to an EnKF may improve its accuracy and/or enhance its stability against filter divergence, especially in the small ensemble scenario.

Notations and assumptions

Notations:
- \( X_k^p \): analysis ensemble at time \( k \), with \( N \) members
- \( x_k^p \): analysis (sample) mean of \( X_k^p \)
- \( x_k^p \): true state system (truth)
- \( E y_k \): observation vector
- \( cov (E) \): covariance
- \( dim (y_k) = n \) and \( dim (y_k) = p \)
- \( R_k \): innovation vector
- \( H_k \): linear observation operator
- \( p \leq n \)

Residual nudging

Some thoughts
- \( \| \mathbf{r}_k^p \| = \| H_k \mathbf{x}_k^p - H_k \mathbf{x}_k^c \| + \| \mathbf{v}_k \| \)
- \( \mathbf{r}_k^p \) for a reasonable estimate one might expect that \( \| H_k \mathbf{x}_k^p - H_k \mathbf{x}_k^c \| \)
- \( \| \mathbf{E} \mathbf{v}_k \| \leq \beta \sqrt{\mathbf{R}_k} \), \( \mathbf{R}_k \) is a real scalar parameter, called noise level hereafter

Implementation:
- If \( \mathbf{r}_k^p \leq \beta \sqrt{\mathbf{R}_k} \), keep the original estimate \( x_k^p \) (equivalent to letting \( \mathbf{c}_k = 1 \) below)
- Otherwise replace \( x_k^p \) by \( x_k^p = \mathbf{c}_k x_k^p = (1 - \beta) x_k^p + \beta \mathbf{r}_k^p \), where \( \beta = \frac{\sqrt{\mathbf{R}_k}}{\| \mathbf{r}_k^p \|} \)

Experiment results with a linear scalar model
- dynamical model: \( x_{k+1} = 0.9 x_k + u_k \), with \( u_k \sim N(0, 1) \)
- observation model: \( y_k = x_k + v_k \), with \( v_k \sim N(0, 1) \)

Experiment results with the Lorenz 96 model
- dynamical model: \( \frac{d x}{dt} = (x_{k+1} - x_k) x_k + f \)
- observation model (every 4 \( \times 0.05 \) time units): \( y_k = [x_k, x_{k+1}, \ldots, x_{k+39}, x_{k+40}] + \mathbf{v}_k \), with \( d \) being an integer, \( J = floor(39/d) \) and \( \mathbf{v}_k \sim N(0, 1) \)

Experiment Results w.r.t the ensemble adjustment Kalman filter (EAKF, Anderson, 2001) and the EAKF with residual nudging (EAKF-RN):
- Results with different observation operators \( d = 1, 2, 4, 8 \)
- Results with different noise level
- Nonlinearity in the observation operators was not addressed yet (Luo and Hotiel, 2012). A possible strategy is to linearize a nonlinear observation operator and conduct iterative searching to find a corresponding observation inversion
- Application of residual nudging to other types of EnKFs can be done in a similar way, while the extension in terms of other data assimilation methods with residual nudging (DARN) may be also possible (to be reported elsewhere)

Remaining issues and future works

References

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