Ensemble Kalman filtering with residual nudging

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Summary

Covariance inflation and localization are two important techniques that are used to improve the performance of the ensemble Kalman filter (EnKF) by adjusting the sample covariances of the estimates in the state space. In this work an additional auxiliary technique, called residual nudging, is proposed to monitor and, if necessary, adjust the residual norms of state estimates in the observation space. In an EnKF with residual nudging, if the residual norm of an analysis is larger than a pre-specified value, then the analysis is replaced by a new one whose residual norm is no larger than a pre-specified value. Otherwise the analysis is considered as a reasonable estimate and no change is made. A rule for choosing the pre-specified value is suggested. Based on this rule, the corresponding new state estimates are explicitly derived in case of linear observations. Numerical experiments in the 40-dimensional Lorenz 96 model show that introducing residual nudging to an EnKF may improve its accuracy and/or enhance its stability against filter divergence, especially in the small ensemble scenario.

Notations and assumptions

Notations:

- $\mathbf{X}_k^a = \{\mathbf{x}_{k,i}^a\}_{i=1}^N$: analysis ensemble at time k, with N members
- $\hat{\mathbf{x}}_k^a$: analysis (sample) mean of $\mathbf{X}_k^a$; $\mathbf{x}_k^{tr}$: true system state (truth)
- Observation system: $\mathbf{y}_k = \mathbf{H}_k(\mathbf{x}_k) + \mathbf{v}_k$, with $\mathbb{E}(\mathbf{v}_k) = \mathbf{0}$ and $\mathbf{cov}(\mathbf{v}_k) = \mathbf{R}_k$
- $\dim(\mathbf{x}_k) = n$ and $\dim(\mathbf{y}_k) = p$
- Residual $\mathbf{r}_k^a \equiv \mathbf{H}_k(\hat{\mathbf{x}}_k^a) - \mathbf{y}_k = [\mathbf{H}_k(\hat{\mathbf{x}}_k^a) - \mathbf{H}_k(\mathbf{x}_k^{tr})] - \mathbf{v}_k$
- Residual norm $\|\mathbf{r}_k^a\| \equiv \sqrt{(\mathbf{r}_k^a)^T \mathbf{R}_k^{-1} \mathbf{r}_k^a}$

Assumptions:

- $\mathbf{H}_k$ is a linear observation operator, with full row rank
- $p \leqslant n$

Residual nudging

Some thoughts:

- $\|\mathbf{r}_k^a\| \leqslant \|\mathbf{H}_k(\hat{\mathbf{x}}_k^a) - \mathbf{H}_k(\mathbf{x}_k^{tr})\| + \|\mathbf{v}_k\|$
- For a reasonable estimate one might expect that $\|\mathbf{H}_k(\hat{\mathbf{x}}_k^a) - \mathbf{H}_k(\mathbf{x}_k^{tr})\|$ is in the order of $\|\mathbf{v}_k\|$ (or less)
- $(\mathbb{E}\|\mathbf{v}_k\|)^2 \leqslant \mathbb{E}\|\mathbf{v}_k\|^2 = \mathrm{trace}(\mathbb{E}(\mathbf{v}_k \mathbf{v}_k^T)\mathbf{R}_k^{-1}) = p$, i.e., $\mathbb{E}\|\mathbf{v}_k\| \leqslant \sqrt{p}$

Objective: Make the residual norm of the state estimate no larger than $\beta\sqrt{p}$, where $\beta \geqslant 0$ is a real scalar parameter, called noise level hereafter

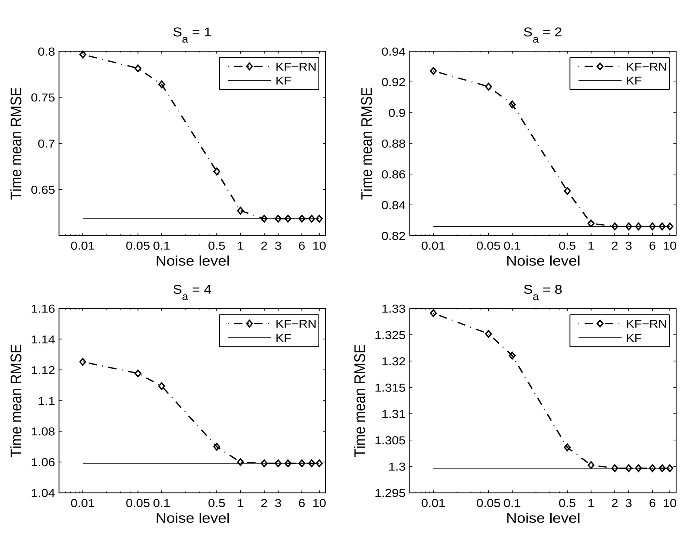
Implementation:

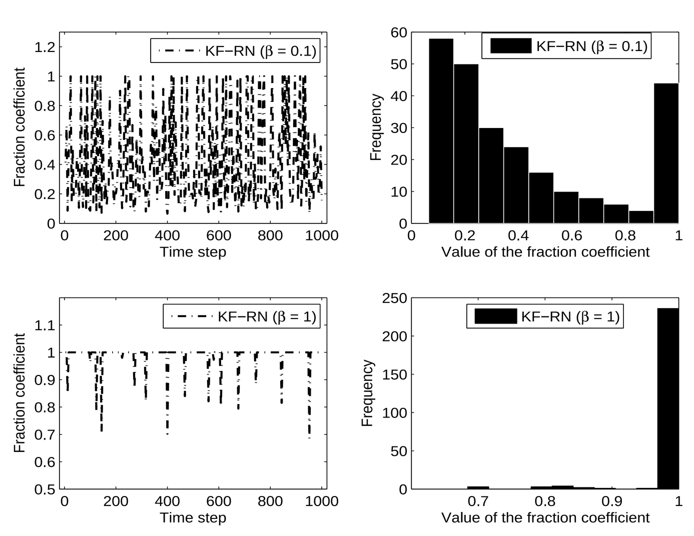
- if $\|\mathbf{r}_k^a\| \leqslant \beta\sqrt{p}$, keep the original estimate $\hat{\mathbf{x}}_k^a$ (equivalent to letting $c_k = 1$ below)
- otherwise replace $\hat{\mathbf{x}}_k^a$ by $\tilde{\mathbf{x}}_k^a = c_k\,\hat{\mathbf{x}}_k^a + (1 - c_k)\,\mathbf{x}_k^o$, where
  - $c_k = \beta\sqrt{p}/\|\hat{\mathbf{r}}_k^a\|$ (more generally, $c_k = \min(1, \beta\sqrt{p}/\|\hat{\mathbf{r}}_k^a\|)$) is called the fraction coefficient
  - $\mathbf{x}_k^o$ is a solution of the equation $\mathbf{H}_k \mathbf{x}_k = \mathbf{y}_k$, e.g., in the form of $\mathbf{x}_k^o = \mathbf{R}_k^{-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{R}_k^{-1}\mathbf{H}_k^T)^{-1}\mathbf{y}_k$

  such that $\tilde{\mathbf{r}}_k^a = \mathbf{H}_k\tilde{\mathbf{x}}_k^a - \mathbf{y}_k = c_k \mathbf{r}_k^a$ and $\|\tilde{\mathbf{r}}_k^a\| = \beta\sqrt{p}$
- in either case, no change of the analysis (sample) covariance is made

Experiment results with a linear scalar model

- dynamical model: $x_{k+1} = 0.9\,x_k + u_k$, with $u_k \sim N(u_k : 0, 1)$
- observation model: $y_k = x_k + v_k$, with $v_k \sim N(v_k : 0, 1)$





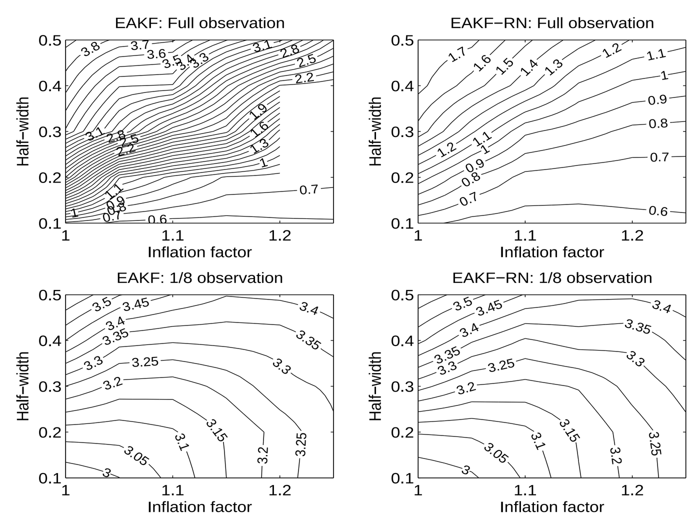
S_a is the time interval between consecutive observations; KF: Kalman filter; KF-RN: KF with residual nudging.

Experiment results with the Lorenz 96 model

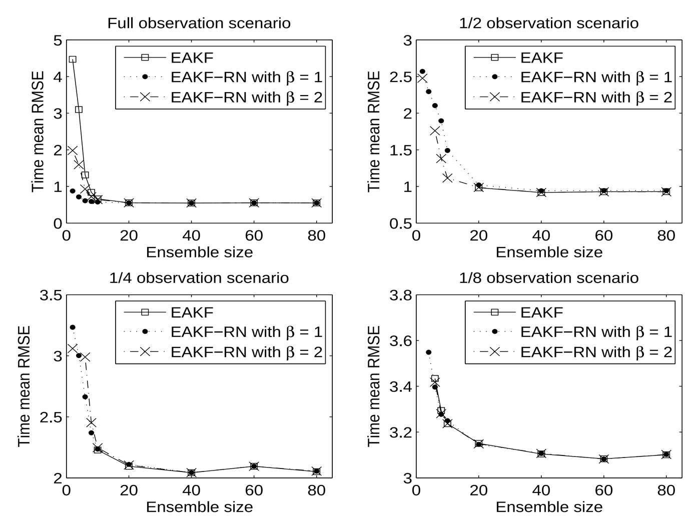
- dynamical model: $\frac{dx_i}{dt} = (x_{i+1} - x_{i-2})\,x_{i-1} - x_i + 8$, $i = 1, \cdots, 40$
- observation model (every 4×0.05 time units): $\mathbf{y}_k = [x_{k,1}, x_{k,1+d}, \cdots, x_{k,1+Jd}]^T + \mathbf{v}_k$, with d being an integer, $J = \mathrm{floor}(39/d)$ and $\mathbf{v}_k \sim N(\mathbf{v}_k : \mathbf{0}, \mathbf{I}_{J+1})$

Experiment Results w.r.t the ensemble adjustment Kalman filter (EAKF,Anderson, 2001) and the EAKF with residual nudging (EAKF-RN):

- Results with different observation operators ($d = 1, 2, 4, 8$)



RMSEs of the EAKF and EAKF-RN, as functions of inflation factor λ and half-width l_c of covariance localization, in the full ($d = 1$) and 1/8 ($d = 8$) observation scenarios



Time mean RMSEs of the EAKF and the EAKF-RN, as functions of the ensemble size in different observation scenarios

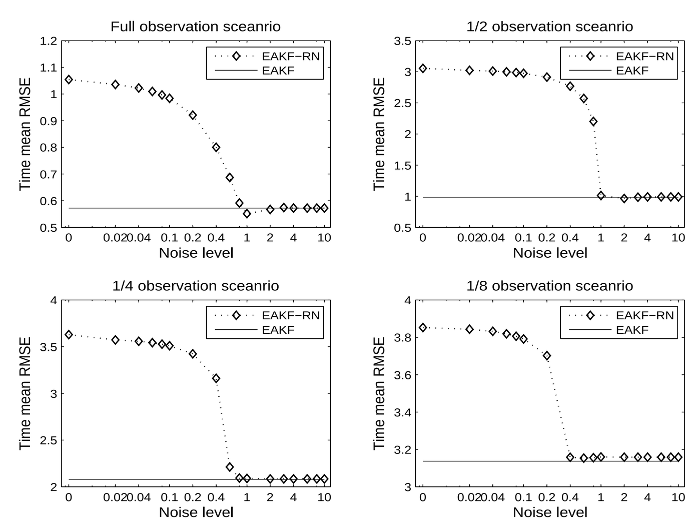
EAKF	$l_c = 0.1$	$l_c = 0.2$	$l_c = 0.3$	$l_c = 0.4$	$l_c = 0.5$
$\lambda = 1.00$	1.0721	Div	Div	Div	Div
$\lambda = 1.05$	1.0091	Div	Div	Div	Div
$\lambda = 1.10$	0.9789	Div	Div	Div	Div
$\lambda = 1.15$	0.9662	Div	Div	Div	Div
$\lambda = 1.20$	0.9515	Div	Div	Div	Div
$\lambda = 1.25$	0.9623	Div	Div	Div	Div
EAKF-RN	$l_c = 0.1$	$l_c = 0.2$	$l_c = 0.3$	$l_c = 0.4$	$l_c = 0.5$
$\lambda = 1.00$	1.0325	1.8256	2.1099	2.2734	2.2964
$\lambda = 1.05$	1.0051	1.4072	1.9879	2.1821	2.2468
$\lambda = 1.10$	0.9598	1.2313	1.8517	2.0342	2.1742
$\lambda = 1.15$	0.9673	1.2024	1.6507	1.9317	2.0953
$\lambda = 1.20$	0.9474	1.1788	1.5776	1.9059	2.0806
$\lambda = 1.25$	0.9650	1.1856	1.5315	1.7778	2.0071

RMSEs of the EAKF and EAKF-RN, as functions of inflation factor and half-width of covariance localization, in the 1/2 ($d = 2$) observation scenario

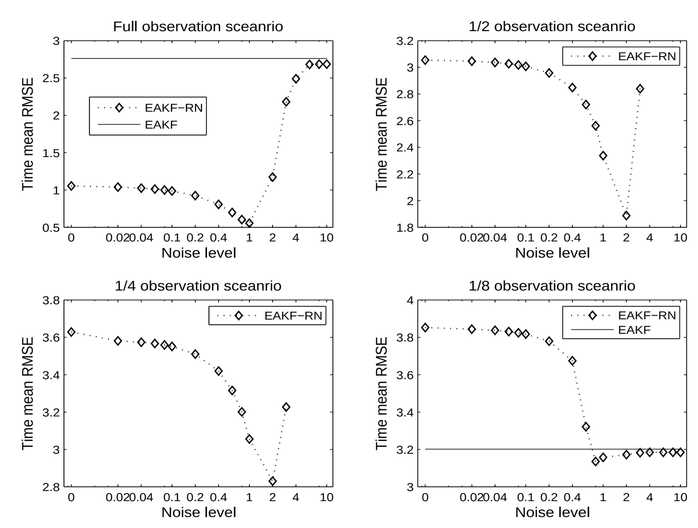
EAKF	$l_c = 0.1$	$l_c = 0.2$	$l_c = 0.3$	$l_c = 0.4$	$l_c = 0.5$
$\lambda = 1.00$	2.0685	Div	Div	Div	Div
$\lambda = 1.05$	1.9908	Div	Div	Div	Div
$\lambda = 1.10$	2.0223	2.3014	Div	Div	Div
$\lambda = 1.15$	2.0819	2.2174	2.9502	Div	Div
$\lambda = 1.20$	2.1903	2.1839	2.7534	Div	Div
$\lambda = 1.25$	2.3586	2.2596	2.6413	Div	Div
EAKF-RN	$l_c = 0.1$	$l_c = 0.2$	$l_c = 0.3$	$l_c = 0.4$	$l_c = 0.5$
$\lambda = 1.00$	2.0840	2.6099	3.0267	3.0453	3.0469
$\lambda = 1.05$	2.0042	2.3341	2.8493	3.0573	3.1015
$\lambda = 1.10$	1.9860	2.2976	2.8154	3.0527	3.1251
$\lambda = 1.15$	2.0766	2.2389	2.7737	3.1247	3.2583
$\lambda = 1.20$	2.1886	2.2312	2.6566	3.0992	3.2340
$\lambda = 1.25$	2.3436	2.2352	2.6168	3.0977	3.2897

RMSEs of the EAKF and EAKF-RN, as functions of inflation factor and half-width of covariance localization, in the 1/4 ($d = 4$) observation scenario

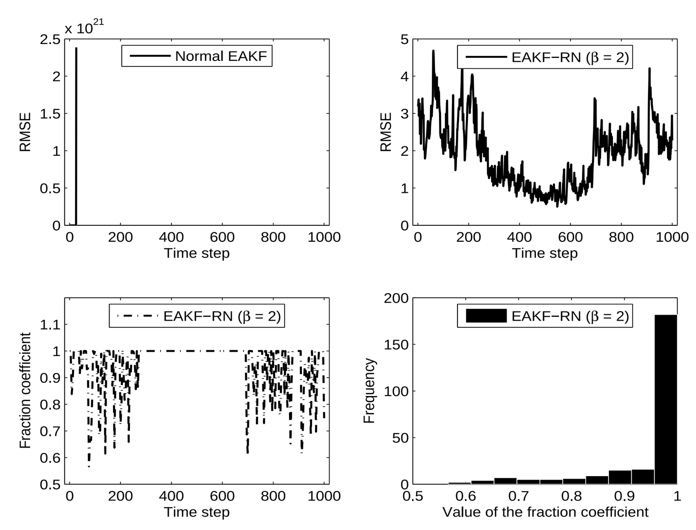
- Results with different noise level



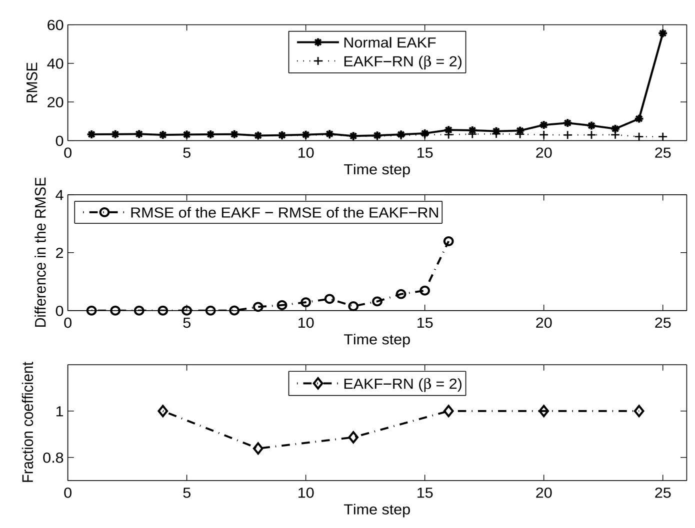
Time mean RMSEs as functions of the noise level in different observation scenarios, with $\lambda = 1.15$ and $l_c = 0.1$



As in the left figure, but with $\lambda = 1.05$ and $l_c = 0.3$ for both the normal EAKF and the EAKF-RN



Upper left: sample time series of the RMSE of the normal EAKF in the 1/2 observation scenario; Upper right: corresponding sample time series of the RMSE of the EAKF-RN ($\beta = 2$); Lower left: corresponding fraction coefficient c_k in the EAKF-RN ($\beta = 2$); Lower right: corresponding histogram of c_k



Upper: the RMSE of the EAKF and EAKF-RN ($\beta = 2$) between the time instant $k = 1$ and $k = 25$; Middle: difference in the RMSE (= RMSE of the EAKF - RMSE of the EAKF-RN) between $k = 1$ and $k = 16$; Lower: the fraction coefficient of the EAKF-RN ($\beta = 2$) between $k = 1$ and $k = 25$

Remaining issues and future works

- Nonlinearity in the observation operators was not addressed yet (Luo and Hoteit, 2012). A possible strategy is to linearize a nonlinear observation operator and conduct iterative searching to find a corresponding observation inversion
- Application of residual nudging to other types of EnKFs can be done in a similar way, while the extension in terms of other data assimilation methods with residual nudging (DARN) may be also possible (to be reported elsewhere)

References

Anderson, J. L., 2001: An ensemble adjustment Kalman filter for data assimilation. *Mon. Wea. Rev.*, **129**, 2884–2903.

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