Particle Filters in High-Dimensional Systems

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Introduction

- With sufficient particles or model runs, the Sequential Importance Resampling (SIR) particle filter gives a representation of the full posterior pdf.

- In high dimensional systems, more model runs would be required than is realistically feasible.

- SIR particle filter must be adapted to ensure that the model runs available all contribute significant information on the posterior pdf.

- This presentation will look at the equivalent weights particle filter.
Equivalent weights particle filter in high-dimensional systems

- Every other variable is observed
- 32 particles
- Observations every 50 timesteps
- 1200 time steps
Equivalent weights particle filter in high-dimensional systems

True model state (timestep 1150)

One particle (of 32)
The equivalent weights particle filter – relaxation proposal density

\[ w_i^{n-1} = \prod_{j=1}^{n-1} \frac{p(x_i^j | x_i^{j-1})}{q(x_i^j | x_i^{j-1}, y^n)} \]

\[ x_i^j = f(x_i^{j-1}) + d\beta_i^j \]

\[ x_i^j = f(x_i^{j-1}) + d\beta_i^j + B(\tau)(y^n - h(x_i^{j-1})) \]
Proposal density can ensure equivalent weights for majority of particles

\[ w_{i}^{n-1} = \prod_{j=1}^{n-1} \frac{p(x_i^{j} | x_i^{j-1})}{q(x_i^{j} | x_i^{j-1}, y^n)} \]

\[ w_{i}^{j} = \frac{p(x_i^{j} | x_i^{j-1})}{q(x_i^{j} | x_i^{j-1}, y^n)} \]

\[ x_i^{j} = f(x_i^{j-1}) + d\beta_i^j \quad \text{and} \quad x_i^{j} = f(x_i^{j-1}) + d\beta_i^j + B(\tau)(y^n - h(x_i^{j-1})) \]
Effect on model balances

To what extent does the additional movement determined by both the relaxation term and the equivalent weights movement effect model balances?

Relative movement in the Barotropic Vorticity equation

<table>
<thead>
<tr>
<th>%</th>
<th>$E_{i,n}[f(x_i^{n-1})]$</th>
<th>$E_{i,n}[d\beta_i^{n-1}]$</th>
<th>$E_{i,n}[B(\tau)(y^n - Hx_i^{n-1})]$</th>
<th>$E_{i,n}$[equ.weights]</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>11.43</td>
<td>1.28</td>
<td>0.09</td>
<td>11.89</td>
</tr>
<tr>
<td>80</td>
<td>11.40</td>
<td>1.28</td>
<td>0.09</td>
<td>13.94</td>
</tr>
<tr>
<td>90</td>
<td>11.29</td>
<td>1.28</td>
<td>0.13</td>
<td>25.83</td>
</tr>
<tr>
<td>100</td>
<td>11.08</td>
<td>1.28</td>
<td>0.22</td>
<td>53.95</td>
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</table>
**Primitive equation model**

**h field**

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**Primitive equations:**

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -g \frac{\partial h}{\partial x}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -g \frac{\partial h}{\partial x}
\]

**Continuity equation:**

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) = 0
\]

**Observations:**

- ‘h’ only, every 3rd grid point ~ 30km
- every 10 days
- error of 5m
- corresponds to 2.5cm of sea-surface height

**Model error:**

- balanced model error
- scaled to be 5–10% of deterministic movement in u and v values which leads to values of ~1% in h field
Performance of equivalent weights particle filter

True model state

Mean of particles

-32 particles
-100% of particles retained
Relaxation and equivalent weights balanced movement

Relaxation model equation:

\[ x_i^j = f(x_i^{j-1}) + d\beta_i^j + B(\tau)(y^n - h(x_i^{j-1})) \]

\[ d\beta_i^j = Q^{1/2} \xi_i^j \quad \text{where} \quad \xi_i^j \sim N(0, I) \]

\[ B(\tau) = \tau bQH^TR^{-1} \]

Equivalent weights:

\[ x_i^n = f(x_i^{n-1}) + \alpha_i QH^T(HQH^T + R)^{-1}(y^n - Hf(x_i^{n-1})) + \tilde{d}\beta_i^n \]

\[ Q = U\tilde{U}Q\tilde{\psi}\tilde{U}^TU^T \]

\[ U : \psi \to h, u, v \]

\[ \tilde{U} : \tilde{\psi} \to \psi \]
Balanced and unbalanced motion

\[ \mathbf{u}_{\text{ageostrophic}} = \mathbf{u} - \mathbf{u}_{\text{geostrophic}} \]

\[ |\mathbf{u}| > 0.1 \]
Gravity waves

20 day run – samples every 3hrs
Gravity waves

Gravity wave dispersion relation:

$$\omega^2 = f_0^2 + g' H (k^2 + l^2)$$

$$= f_0^2 + g' H (\kappa^2)$$
### Gravity waves results

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Random error</th>
<th>Relaxation</th>
<th>Equivalent weights</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Without</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gravity</td>
<td>5.385 x 10^{13}</td>
<td>5.392 x 10^{13}</td>
<td>5.062 x 10^{13}</td>
<td>5.433 x 10^{13}</td>
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<tr>
<td>Above</td>
<td>9.583 x 10^{12}</td>
<td>9.553 x 10^{12}</td>
<td>1.523 x 10^{13}</td>
<td>1.060 x 10^{13}</td>
</tr>
<tr>
<td>Below</td>
<td>2.235 x 10^{15}</td>
<td>2.223 x 10^{15}</td>
<td>2.492 x 10^{15}</td>
<td>3.003 x 10^{15}</td>
</tr>
<tr>
<td><strong>With</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gravity</td>
<td>1.143 x 10^{14}</td>
<td>1.141 x 10^{14}</td>
<td>1.110 x 10^{14}</td>
<td>1.202 x 10^{14}</td>
</tr>
<tr>
<td>Above</td>
<td>9.583 x 10^{12}</td>
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<td>1.060 x 10^{13}</td>
</tr>
<tr>
<td>Below</td>
<td>2.175 x 10^{15}</td>
<td>2.163 x 10^{15}</td>
<td>2.432 x 10^{15}</td>
<td>2.937 x 10^{15}</td>
</tr>
</tbody>
</table>

- **Relaxation:**
  - Suppression of gravity waves.
  - Increase is in high frequency noise
    (would normally die, but energy in these modes is reinforced every time step)

- **Equivalent weights:**
  - mainly affects low frequencies (40% increase)
  - gravity waves increase by 8%
Conclusions

Two primary questions that need to be answered before consideration for actual geophysical applications:

1. Does the scheme perform well in high-dimensional systems as the number and distribution of observations changes?

2. What effect does the relaxation term and equivalent weights movement have on model balances and gravity waves?

First question has been looked at in detail with reference to the barotropic vorticity equation.

This presentation has shown that there does not appear to be any large issues related to the second question.
Thank you and any questions?