Weighted Ensemble Transform Kalman Filter for Image Assimilation

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Fluminance

Introduction

Geophysical flow analysis

- Satellite images have a great potential for submesoscale analysis
- Image data are only poorly taken into account in data assimilation

Objective

- Explore stochastic filtering techniques for flow reconstruction directly from image sequences

Stochastic filtering in a non linear setting

Principle

- Given $d\mathbf{x}_t = \mathbf{M}(\mathbf{x}_t)dt + \sigma(t)d\mathbf{B}_t$ and $\mathbf{y}_k = \mathbf{H}(\mathbf{x}_k) + \gamma_k$
- Estimate the pdf $p(\mathbf{x}_{0:k}|\mathbf{y}_{1:k}) = p(\mathbf{x}_{0:k-1}|\mathbf{y}_{1:k-1})\frac{p(\mathbf{y}_k|\mathbf{x}_{t=k})p(\mathbf{x}_{t=k}|\mathbf{x}_{k-1})}{p(\mathbf{y}_k|\mathbf{y}_{1:k-1})}$

Gaussian linear model: Kalman Filtering

$$\mathbb{E}(\mathbf{x}_k|\mathbf{y}_{1:k}) = \overline{\mathbf{x}}_k^a = \overline{\mathbf{x}}_{k|k-1} + \mathbf{K}(\mathbf{y}_t - \mathbf{H}\overline{\mathbf{x}}_{k|k-1}),$$
$$\mathbf{K} = \mathbf{\Sigma}_{k|k-1}\mathbf{H}^T(\mathbf{H}\mathbf{\Sigma}_{k|k-1}\mathbf{H}^T + \mathbf{R})^{-1}$$
$$\mathbb{E}((\mathbf{x}_k - \mathbf{x}_k^a)(\mathbf{x}_k - \mathbf{x}_k^a)^T|\mathbf{y}_{1:k}) = \mathbf{P}_k^a = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{\Sigma}_{k|k-1}$$

High dimensional extension: Ensemble Kalman Filtering [Evensen 94]

Kalman updates computed from a set of samples $\mathbf{x}_t^{(i)},\ i = 1, \ldots, N$

Particle Filter

Non-linear dynamics and observations

- $p(\mathbf{x}_k|\mathbf{y}_{1:k}) \simeq \sum_i \omega_k^{(i)} \delta_{\mathbf{x}_k^{(i)}}$
- prediction step (importance distribution sampling π)

$$\mathbf{x}_k^{(i)} \sim \pi(\mathbf{x}_{0:k}|\mathbf{y}_{1:k}) = \pi(\mathbf{x}_{0:k-1}|\mathbf{y}_{1:k-1})\pi(\mathbf{x}_k|\mathbf{y}_{1:k}, \mathbf{x}_{0:k-1})$$

- correction step

$$\omega_k^{(i)} \propto \omega_{k-1}^{(i)} \frac{p(\mathbf{y}_k|\mathbf{x}_k^{(i)})p(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)}|\mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})}$$

Ensemble Kalman filter extension

Importance distribution

- Bootstrap filter

$$\pi(\mathbf{x}_k|\mathbf{x}^{(i)}_{0:k-1}, \mathbf{y}_{1:t}) = p(\mathbf{x}_k|\mathbf{x}^{(i)}_{k-1}) \Rightarrow w^{(i)}_k \propto \omega^{(i)}_{k-1} p(\mathbf{y}_k|\mathbf{x}^{(i)}_k)$$

$\Rightarrow$ strong limitation in high dimensional space

- Ensemble Kalman proposal distribution [Papadakis et al 10]

$$\pi(\mathbf{x}_k|\mathbf{x}^{(i)}_{0:k-1}, \mathbf{z}_{1:k}) = p(\mathbf{x}_k|\mathbf{x}^{(i)}_{k-1}, \mathbf{y}_k) \approx \mathcal{N}(\overline{\mathbf{x}}^a_k, (\mathbb{I} - \mathbf{K}^e\mathbf{H})\boldsymbol{\Sigma}^e_{k|k-1})$$

$$\omega^{(i)}_k \propto \omega^{(i)}_{k-1} \frac{p(\mathbf{y}_k|\mathbf{x}^{(i)}_k) p(\mathbf{x}^{(i)}_k|\mathbf{x}^{(i)}_{k-1})}{\mathcal{N}\left(\mathbf{x}^{(i)}_k - \overline{\mathbf{x}}^a_k; 0, \mathbf{P}^a_k\right)}$$

where with linear observation operator

$$(N-1)\mathbf{P}^a_k = \mathbf{X}^f_k {\mathbf{X}^f_k}^T - \mathbf{X}^f_k {\mathbf{X}^f_k}^T \mathbf{H}^T (\mathbf{H}\mathbf{X}^f_k {\mathbf{X}^f_k}^T \mathbf{H}^T + \tilde{\mathbf{R}})^{-1} \mathbf{H}\mathbf{X}^f_k {\mathbf{X}^f_k}^T$$

Ensemble Kalman filter extension

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$$\omega^{(i)}_k \propto \omega^{(i)}_{k-1} \frac{p(\mathbf{y}_k|\mathbf{x}^{(i)}_k)p(\mathbf{x}^{(i)}_k|\mathbf{x}^{(i)}_{k-1})}{\mathcal{N}\left(\mathbf{x}^{(i)}_k - \overline{\mathbf{x}}^a_k; 0, \mathbf{P}^a_k\right)}$$

with nonlinear observation operator

$$(N-1)\mathbf{P}^a_k = \mathbf{X}^f_k {\mathbf{X}^f_k}^T - \mathbf{X}^f_k \mathbf{H}(\mathbf{X}^f_k)^T (\mathbf{H}(\mathbf{X}^f_k)\mathbf{H}(\mathbf{X}^f_k)^T + \tilde{\mathbf{R}})^{-1} \mathbf{H}(\mathbf{X}^f_k) {\mathbf{X}^f_k}^T,$$

Ensemble Kalman filter extension

Ensemble Transform Kalman filter [Bishop et al 01, Tipett et al 03, Wang et al 04, ...]

- Provides an estimate of

$$\mathbf{P}_k^a = \mathbf{X}_k^f \mathbf{A}_k \mathbf{A}_k^T {\mathbf{X}_k^f}^T$$
$$\mathbf{A}_k \mathbf{A}_k^T = (\mathbf{I} + \mathbf{H}(\mathbf{X}_k^f)^T \tilde{\mathbf{R}}^{-1} \mathbf{H}(\mathbf{X}_k^f))^{-1}$$

- Eigenvalue decomposition of $\mathbf{H}(\mathbf{X}_k^f)^T \tilde{\mathbf{R}}^{-1} \mathbf{H}(\mathbf{X}_k^f) = \mathbf{U}_k \mathbf{\Lambda}_k \mathbf{U}_k^T$
- Yields the mean preserving square root

$$\mathbf{A} = \mathbf{U}_k (\mathbf{I} + \mathbf{\Lambda}_k)^{-1/2} \mathbf{U}_k^T$$

- Analysis perturbation ensemble and mean update

$$\mathbf{X}_k^a = \mathbf{X}_k^f \mathbf{A}_k, \text{ and } \bar{\mathbf{x}}_k^a = \bar{\mathbf{x}}_k^f + (N-1)^{-1} \mathbf{X}_k^f \mathbf{A}_k \mathbf{A}_k^T \mathbf{H}(\mathbf{X}_k^f)^T \mathbf{R}^{-1} (\mathbf{y}_k - \mathbf{H}(\bar{\mathbf{x}}_k^f))$$

Ensemble Kalman filter extension

Weighted Ensemble Transform Kalman filter

At iteration k

- Get $\mathbf{x}_k^{a,(i)}$ from the forecast and analysis steps of ETKF:
  - Forecast: for all $i = 1, \ldots, N$, simulate $\mathbf{x}_k^{f,(i)}$ from $\mathbf{x}_{k-1}^{a,(i)}$
  - Analysis: Compute $\mathbf{x}_k^{a,(i)}$ from ETKF update.
- Compute weights $\omega_k^{(i)} \propto p(\mathbf{y}_k|\mathbf{x}_k^{a,(i)})$.
- Resample particles to obtain $\{\mathbf{x}_k^{(i)}, i = 1, \ldots, N\}$ and set $\omega_k^{(i)} = \frac{1}{N}$ for all $i = 1, \ldots, N$.

Vorticity recovering from image data

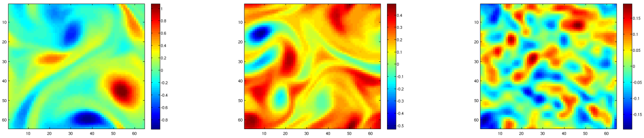
Filtering system

- Dynamics

$$d\xi + \nabla\xi \cdot \mathbf{w} dt = \frac{1}{Re}\Delta\xi dt + \sigma_Q dW,$$

- dW isotropic Gaussian field

$$Q(\mathbf{r},\tau) = \mathbb{E}(dW(\mathbf{x},t)dW(\mathbf{x}+\mathbf{r},t+\tau)) = g_\lambda(\mathbf{r})dt\delta(\tau),$$



Vorticity recovering from image data

Filtering system

- Measurements

  1) Local motion mesurements and motion uncertainties

$$\tilde{\xi}_k = \xi + \boldsymbol{\gamma}_k$$

$$p(\tilde{\boldsymbol{\xi}}_k | \boldsymbol{\xi}_k^{(i)}) \propto \exp\left(-\frac{1}{2}(\tilde{\boldsymbol{\xi}}_k - \boldsymbol{\xi}_k^{(i)})^T \mathbf{R}^{-1}(\tilde{\boldsymbol{\xi}}_k - \boldsymbol{\xi}_k^{(i)})\right)$$

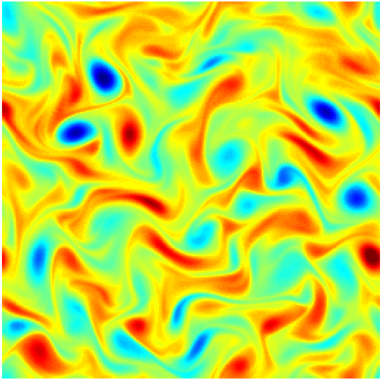
  2) Image reconstruction error

$$\mathbf{I}(\mathbf{x}, k) = \mathbf{I}(\mathbf{x} + \mathbf{d}_{k+1}(\mathbf{x}), k+1) + \boldsymbol{\eta}_k$$

$$p(\mathbf{I}_k | \boldsymbol{\xi}_k^{(i)}) \propto \exp\left(-\int_\Omega \frac{1}{\sigma^2(\mathbf{x})}\left(I(\mathbf{x}, k) - I(\mathbf{x} + \mathbf{d}^{(i)}(\mathbf{x}), k+1)\right)^2\right) d\mathbf{x}.$$

$$\sigma_k^2(\mathbf{x}) = \frac{1}{N-1}\sum_{i=1}^{N}(I(\mathbf{x} + \mathbf{d}^{(i)}(\mathbf{x}), k+1) - \overline{I_d}(\mathbf{x}, k+1))^2 + \epsilon,$$

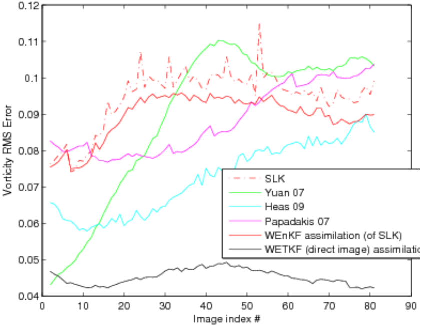
Results: 2D DNS sequence



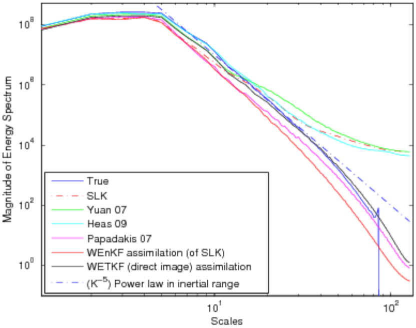
passive scalar

vorticity

Results: 2D DNS sequence - RMSE - Energy Spectrum

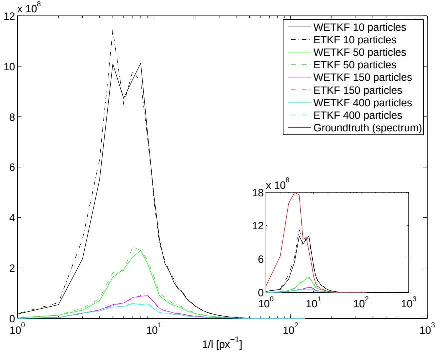


RMSE vorticity

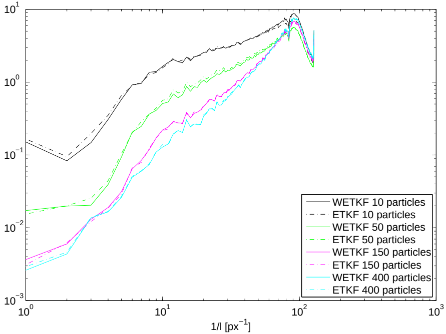


Energy Spectrum

Results: 2D DNS sequence - Error energy Spectrum - different numbers of particles



Error Energy spectrum (scale repr.)

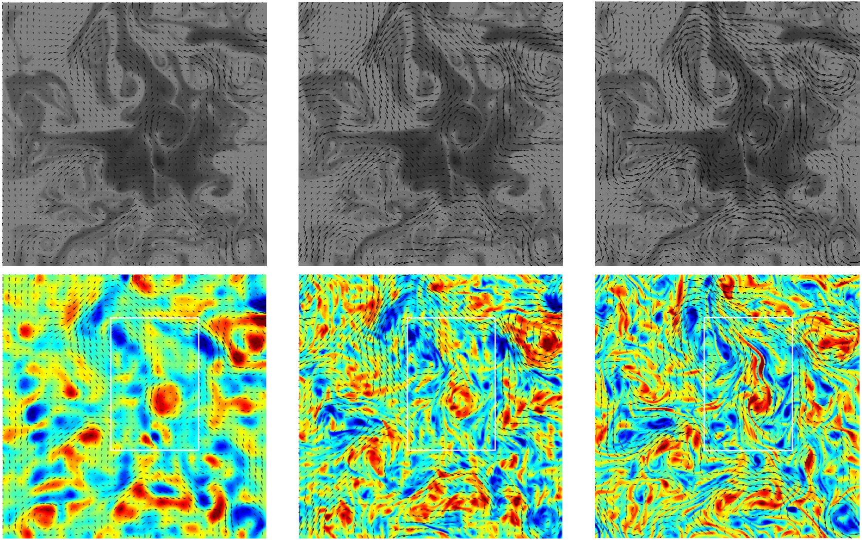


Error Energy Spectrum/ Energy spectrum

Results: 2D DNS sequence

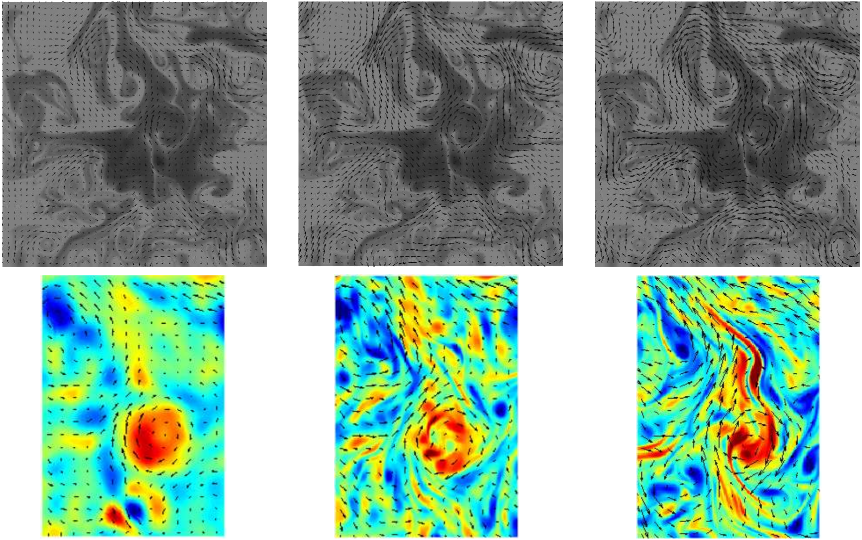
Filtering results (initialization with local motion estimates)

Results: 2D experimental turbulence



w_{obs} | WETKF (w_{obs}) | WETKF ($\hat{I}_{k+1} - I_k$)

Results: 2D experimental turbulence



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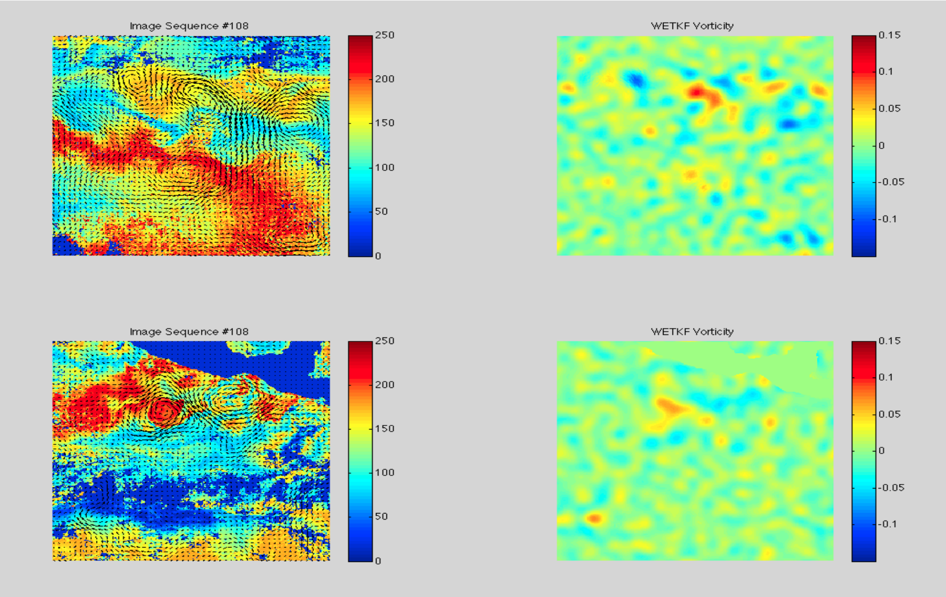
Results: Oceanic SST images



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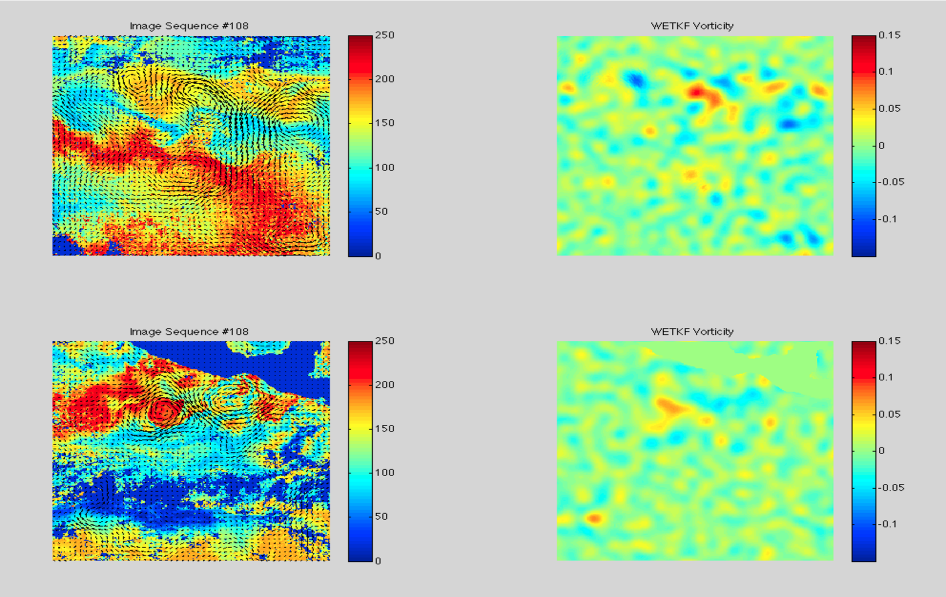

Conclusion

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- Study of mean preserving square root filters as proposal distribution of a particle filter
- Use of a nonlinear image reconstruction operator for the direct assimilation of image data
  $\Rightarrow$ much more efficient than velocity pseudo observations
- ETKF or WETKF needs a relatively high number of particles to get accurate results

Perspectives

- Definition of better norm for the likelihood
- Improve the weight computation
- Extend the approach to realistic oceanic models