Constraining overestimation of error covariances in ensemble Kalman filters

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Toulouse, November 13th, 2012
Constraining overestimation of error covariances in ensemble Kalman filters.

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Observational Data — Forecast Model

Data Assimilation

Climatological Information
What is the effect of the sparsity of observations?

- The obvious: We don’t have much information
- Overestimation of error covariances (exacerbated by finite ensemble sizes) (*Whitaker et al. 2009*)
Using climatology on sparse observational grids

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Applications are

- sparse observational networks
- balance
- model error
- controlling catastrophic filter divergence
Using climatology on sparse observational grids

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Applications are

- sparse observational networks
- balance
- model error
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Further applications are

- re-analysis of climate
- when direct observations are not available (mesosphere)
- general slow-fast systems
Sparse observational grids

Our particular perspective here:

Proper (noisy) observations are available for some variables (observables) but not for other unresolved variables, for which only their statistical climatic behaviour such as their variance and their mean is available (pseudo-observables).

Question:

How can the statistical information available for some data which are otherwise not observable, be effectively incorporated into data assimilation to control overestimation?
Assume an $N$-dimensional dynamical system whose dynamics is given by 
\[ \dot{z} = f(z) \] with the state variable $z \in \mathbb{R}^N$ (no model error for now).

**Observables**

Observations $\mathbf{x}_{\text{obs}}$ at observation times $t_n = n\Delta t_{\text{obs}}$

- observation operator $\mathbf{H} : \mathbb{R}^N \rightarrow \mathbb{R}^n$
- $\mathbf{x}_{\text{obs}}(t_i) = \mathbf{H}z(t_i) + \mathbf{r}_{\text{obs}}(t_i)$ with observational noise $\mathbf{r}_{\text{obs}}$
- $\mathbf{r}_{\text{obs}} \sim \mathcal{N}(0, \mathbf{R}_{\text{obs}})$ with error covariance matrix $\mathbf{R}_{\text{obs}}$
Assume an $N$-dimensional dynamical system whose dynamics is given by $\dot{z} = f(z)$ with the state variable $z \in \mathbb{R}^N$ (no model error for now).

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- $x_{\text{obs}}(t_i) = H z(t_i) + r_{\text{obs}}(t_i)$ with observational noise $r_{\text{obs}}$
- $r_{\text{obs}} \sim \mathcal{N}(0, R_{\text{obs}})$ with error covariance matrix $R_{\text{obs}}$

**pseudo-observables**

Assume climatic knowledge about the pseudo-observables $y$ (mean $a_{\text{target}}$ and variance $A_{\text{target}}$)

- pseudo-observation operator $h : \mathbb{R}^N \rightarrow \mathbb{R}^m$
- $R_w$ is the unknown error covariance matrix associated with the pseudo-observables
Assume an \( N \)-dimensional dynamical system whose dynamics is given by \( \dot{z} = f(z) \) with the state variable \( z \in \mathbb{R}^N \) (no model error for now).

**Observables**

Observations \( x_{\text{obs}} \) at observation times \( t_n = n\Delta t_{\text{obs}} \)

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**Pseudo-Observables**

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- Pseudo-observation operator \( h : \mathbb{R}^N \rightarrow \mathbb{R}^m \)
- \( R_w \) is the unknown error covariance matrix associated with the pseudo-observables

**Question:**

How do we choose/find the error covariance matrix \( R_w \)?
The Variance Limiting Kalman Filter (VLKF)

An ensemble (Evensen, 1996) with \( k \) members \( \mathbf{z}_k \)

\[
\mathbf{Z} = [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k] \in \mathbb{R}^{N \times k}
\]

is propagated by the full nonlinear dynamics

\[
\dot{\mathbf{Z}} = F(\mathbf{Z}), \quad \mathbf{Z}(0) = \mathbf{Z}_b.
\]

The ensemble is split into its mean \( \bar{\mathbf{z}} \) and its ensemble deviation matrix \( \mathbf{Z}' \)

**Step 1: Forecast step**

\[
\mathbf{Z}_f = F(\mathbf{Z}_b)
\]

\[
\mathbf{P}_f = \frac{1}{k - 1} \mathbf{Z}'_f(t)[\mathbf{Z}'_f(t)]^T
\]

Remark: \( \mathbf{P}_f(t) \) is rank-deficient for \( k < N \) (\( N \sim 10^9 \) and \( k \sim 100 \))
Step 2: Analysis step

\[ J(z) = \frac{1}{2}(z - z_f)^T P_f^{-1}(z - z_f) + \frac{1}{2}(Hz - Y)^T R^{-1}(Hz - Y) \]

\[ Y = \begin{pmatrix} x_{\text{obs}} \\ a_{\text{target}} \end{pmatrix}, \quad H = \begin{pmatrix} H \\ h \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} R^{-1}_{\text{obs}} & 0 \\ 0 & R_w^{-1} \end{pmatrix} \]
The Variance Limiting Kalman Filter (VLKF)

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y = \begin{pmatrix} x_{\text{obs}} \\ a_{\text{target}} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H \\ h \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} R_{\text{obs}}^{-1} & 0 \\ 0 & R_w^{-1} \end{pmatrix}
\]

\[
\bar{z}_a = \bar{z}_f - \mathcal{K} [Hz_f - Y]
\]

where \( \mathcal{K} = P_f H^T (HP_f H^T + R)^{-1} \)

with the covariance of the analysis

\[
P_a = [I - \mathcal{K}\mathcal{H}] P_f
\]
Step 2: Analysis step

Constraining the variance of the pseudo-observable $h_2$ is done by requiring

$$hP_a h^T = A_{\text{target}}$$

Introducing $P_a^{-1} = P_f^{-1} + H^T R_{\text{obs}}^{-1} H$, we obtain

$$R_w^{-1} = A_{\text{target}}^{-1} - (hP_a h^T)^{-1}$$
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The naive expectation $R_w = A_{target}$ is true only for

$$|\{R_{obs}, P_f\}| \gg |A_{target}|$$
Step 2: Analysis step

Constraining the variance of the pseudo-observable $h_z$ is done by requiring

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Introducing $P_a^{-1} = P_f^{-1} + H^T R_{\text{obs}}^{-1} H$, we obtain

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- The naive expectation $R_w = A_{\text{target}}$ is true only for $\left| \{R_{\text{obs}}, P_f \} \right| \gg \left| A_{\text{target}} \right|$.
- For sufficiently small background error covariance $P_f$, the error covariance $R_w$ is not positive definite (“switch”): Update only overestimating eigendirections with $\left| hP_a h^T \right| > \left| A_{\text{target}} \right|$.
Step 3: Update of the ensemble

The ensemble needs to be consistent with

$$P_a = \frac{1}{k-1} Z'_a \left[Z'_a\right]^T$$

Method of ensemble square root filters:

- **Ensemble transform Kalman filter (EnTKF)** (*Tippett et al 2003*):
  $$Z'_a = Z'_f S_w \text{ with } S_w \in \mathbb{R}^{k \times k}$$

- **Ensemble adjustment Kalman filter (EnAKF)** (*Anderson 2001*):
  $$Z'_a = A Z'_f \text{ with } A \in \mathbb{R}^{N \times N}$$
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Step 4: Update of the forecast

Set \( Z_b = Z_a \) to propagate the ensemble forward again with the full dynamics to the next observation time.
### Summary of VLKF

#### Step 1: Forecast step

\[ \mathbf{Z}_f = F(\mathbf{Z}_b) \]

\[ \mathbf{P}_f = \frac{1}{k-1} \mathbf{Z}'_f(t)[\mathbf{Z}'_f(t)]^T \]

#### Step 2: Analysis step

\[ \bar{\mathbf{z}}_a = \bar{\mathbf{z}}_f - \mathbf{K}_{\text{obs}}(\mathbf{H}\bar{\mathbf{z}}_f - \mathbf{x}_{\text{obs}}) - \mathbf{K}_w(\mathbf{h}\bar{\mathbf{z}}_f - \mathbf{a}_{\text{target}}) \]

\[ \mathbf{K}_{\text{obs}} = \mathbf{P}_f \mathbf{H}^T(\mathbf{H}\mathbf{P}_f \mathbf{H} + \mathbf{R}_{\text{obs}})^{-1}, \quad \mathbf{K}_w = \mathbf{P}_f \mathbf{h}^T(\mathbf{h}\mathbf{P}_f \mathbf{h} + \mathbf{R}_w)^{-1} \]

\[ \mathbf{R}_w^{-1} = \mathbf{A}^{-1}_{\text{target}} - (\mathbf{h}\mathbf{P}_a \mathbf{h}^T)^{-1} \]

#### Step 3: Update of the ensemble

The ensemble needs to be consistent with

\[ \mathbf{P}_a = [\mathbf{I} - \mathbf{K}_{\text{obs}} \mathbf{H} - \mathbf{K}_w \mathbf{h}] \mathbf{P}_f = \frac{1}{k-1} \mathbf{Z}'_a [\mathbf{Z}'_a]^T \]

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Set \( \mathbf{Z}_b = \mathbf{Z}_a \) to propagate the ensemble forward again with the full dynamics to the next observation time.

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Constraining overestimation of error covariances in ensemble K

Toulouse, November 13th 2012
I. Lorenz-96 model: \[ \dot{z}_i = z_{i-1}(z_{i+1} - z_{i-2}) - z_i + F \]

The pseudo-observables contain the prior climatic knowledge: \( a_{\text{target}} = \mu_{\text{clim}} \) and \( A_{\text{target}} = \sigma_{\text{clim}}^2 I \) with \( \mu_{\text{clim}} = 2.34 \) and \( \sigma_{\text{clim}} = 3.6 \) measured from a long time trajectory \( N_{\text{obs}} = 5, \Delta t_{\text{obs}} = 4 \text{ hours}, R_{\text{obs}} = (0.25\sigma_{\text{clim}})^2 I \)
I. Lorenz-96 model: $\dot{z}_i = z_{i-1} (z_{i+1} - z_{i-2}) - z_i + F$

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$\mathbf{a}_{\text{target}} = \mu_{\text{clim}}$ and $\mathbf{A}_{\text{target}} = \sigma_{\text{clim}}^2 \mathbf{I}$ with $\mu_{\text{clim}} = 2.34$ and $\sigma_{\text{clim}} = 3.6$
measured from a long time trajectory
$N_{\text{obs}} = 5, \Delta t_{\text{obs}} = 4$ hours, $\mathbf{R}_{\text{obs}} = (0.25 \sigma_{\text{clim}})^2 \mathbf{I}$
I. Lorenz-96 model

Quantify the skill improvement by the r.m.s error

\[ E = \sqrt{\frac{1}{LD} \sum_{l=1}^{L} \left\| \bar{z}_a(l\Delta t_{\text{obs}}) - z_{\text{truth}}(l\Delta t_{\text{obs}}) \right\|^2} \]

Skill: \( S = \frac{E^E}{E^V} \)

Best performance of VLKF over ETKF for:
- small \( \Delta t_{\text{obs}} \)
- \( N_{\text{obs}} = 4 \)

Constraining overestimation of error covariances in ensemble K  

Toulouse, November 13th 2012
I. Lorenz-96 model

VLKF produces significant skill in sparse observational grids for
- small observation intervals (< 6 hours)
- the larger the observational noise the better
I. Lorenz-96 model

VLKF produces significant skill in sparse observational grids for

- small observation intervals (< 6 hours)
- the larger the observational noise the better

The overestimation of error covariances in sparse networks is a finite ensemble size effect
II. Filter divergence and blow-up with sparse observations

(traditional) filter divergence: Underestimation of error covariance leads to filter trusting its own forecast for sufficiently large $R_{\text{obs}}$ (cf. Ng et al (2011))
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(Traditional) filter divergence: Underestimation of error covariance leads to filter trusting its own forecast for sufficiently large $R_{\text{obs}}$ (cf. Ng et al (2011))

Catastrophic filter divergence: Filter develops machine-infinity blow-up for sufficiently small $R_{\text{obs}}$ (Harlim & Majda (2010); GAG, Mitchell, Reich (2011))

<table>
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<tr>
<th>$N_{\text{obs}}$</th>
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<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
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</tr>
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<tr>
<td>$\Delta \tau_{\text{obs}}$</td>
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ETKF

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Constraining overestimation of error covariances in ensemble K

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II. Genesis of blow-up

We study the 5D Lorenz-96 model

\[ \dot{z}_i = z_{i-1}(z_{i+1} - z_{i-2}) - z_i + F \quad i = 1, \cdots, 5 \]

with negative forcing \( F = -16 \)

Lyapunov exponents: \( \lambda = (2.72, 0.09, -0.09, -1.83, -5.89) \)

Attractor dimension: \( D_{\text{attr}} = 4.15 \)

Decay rate of the autocorrelation: \( \tau_{\text{corr}} \approx 0.14 \)
II. Genesis of blow-up

- stable until $t_1 \approx 10$
- non-tracking episode until $t_2 \approx 13$
- blow-up for $t > 13$
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RMS error

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Finite size effects: Ensemble dimension \((\text{Patil et al (2001)})\)

$$D_{\text{ens}} = \left( \frac{\sum_{i=1}^{k} \sqrt{\mu_i}}{\sum_{i=1}^{k} \mu_i} \right)^2 \in (1, \min(k, D))$$

where $\mu_i$ are eigenvalues of the $k \times k$ covariance matrix $C = X_f^T X_f$
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Finite size effects: Ensemble dimension (Patil et al. (2001))

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\]

where \( \mu_i \) are eigenvalues of the \( k \times k \) covariance matrix

\[
C = X_f^T X_f
\]

Filter pushes analysis off the attractor

\[
\bar{z}_{a,i} = \bar{z}_{f,i} - \frac{P_{f_{i1}}}{P_{f_{11}} + R_{\text{obs}}} \left[ \bar{z}_{f,i} - x_{\text{obs}} \right]
\]

Lyapunov exponents

\[
\lambda = (2.72, 0.09, -0.09, -1.83, -5.89)
\]
II. Filter divergence and blow-up with sparse observations

Blow-up is caused by the forecast scheme (numerical instability):

Take home message:
Catastrophic blow up may be caused by the combination of finite size ensembles and fast attraction towards the attractor when
- there are sparse but accurate observations
- the underlying system has high variance
III. Controlling balance

Modified Lorenz-96 system \( \) (Bergemann & Reich (2010)\)

\[
\dot{x}_j = (1 - \eta) \left( x_{j-1}(x_{j+1} - x_{j-2}) \right) - x_j + F \\
+ \eta \left( x_{j-1} h_{j+1} - x_{j-2} h_{j-1} \right) \\
\varepsilon^2 \ddot{h}_j = -h_j + \alpha^2 \left( h_{j-1} - 2h_j + h_{j+1} \right) + x_j
\]

- fast part is purely dispersive
- nonlinear terms conserve energy

\[
H = \frac{\eta}{2} \sum_{j=1}^{D} \left( \frac{\eta-1}{\eta} x_j^2 + \varepsilon^2 \dot{h}_j^2 + h_j^2 + \alpha^2 (h_{j+1} - h_{j-1})^2 - 2x_j h_j \right)
\]

- approximate slow manifold given by

\[
\mathcal{B}_j(x_j, h_j) = x_j - (h_j - \alpha^2 (h_{j-1} - 2h_j + h_{j+1})) = 0
\]
III. Controlling balance

Initially balanced fields with
\[ B_j(x_j, h_j) = x_j - (h_j - \alpha^2 (h_{j-1} - 2h_j + h_{j+1})) = 0 \]
do not develop unbalanced motion on very long times:
The amount of unbalance (fast energy) can be measured by

\[ \bar{B}(t) = \sqrt{\frac{1}{D} \sum_{j=1}^{D} (x_j - h_j + \alpha^2 (h_{j-1} - 2h_j + h_{j+1}))^2} \]
The filtering procedure can severely disturb balance (cf. Lorenc (2003), Kepert (2009), Greybush et al (2011)) - this is the case with and without localization in sparse observational networks.

Here only $x$ is observed with $N_{\text{obs}} = 2$ with $h = B$ and $a_{\text{target}} = 0$.

Red: ETKF    Blue: DEnKF with localisation
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Red: ETKF    Blue: DEnKF with localisation    Magenta: VLKF
IV. Model error and forecast bias

A major problem of ensemble filters is underdispersiveness \cite{Buizza2005}.

This can be linked to:

- *dynamical model error*: misrepresentation of unresolved subgrid scale processes \cite{Palmer2001}
- *numerical model error*: large errors produced at grid-scale \cite{underestimation} which are controlled by artificial viscosity \cite{underestimation}
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Main question: Can one get away with underdamped forecast models, yet still control the resulting covariance overestimation within the data assimilation procedure?
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  - unrealistic drainage of energy out of the system (*Shutts 2005*)
  - frontogenesis (*Blumen 1990*)
  - large scale statistics depends on the numerically preserved conservation laws (*Thuburn 2008, Dubinkina & Frank 2007, 2010*)

**Main question**: Can one get away with underdamped forecast models, yet still control the resulting covariance overestimation within the data assimilation procedure?

We will use climatological information of the mean and variance.
IV. Model error and forecast bias

\[
\frac{dz_i}{dt} = z_{i-1}(z_{i+1} - z_{i-2}) - \gamma z_i + F
\]

Truth: \( \gamma = 1 \)
Forecast model: \( \gamma < 1 \)

Perfect model case: \( S \approx 1 \) for \( \Delta t_{\text{obs}} \gg 1 \)
Now we will be interested in the case of \( \Delta t_{\text{obs}} \gg 1 \)
IV. Model error and forecast bias

\[ \gamma = 0.5 \]

\[ \Delta t_{\text{obs}} = 24 \text{ hours} \]
Reproducing the truth

\[ \Delta t_{\text{obs}} = 48 \text{ hours} \]
Reproducing the statistics

(Truth  VLKF  ETKF)
IV. Model error and forecast bias

Reproducing the statistics

\[
\langle \bar{z}_a \rangle \\
\langle (\bar{z}_a - \langle \bar{z}_a \rangle)^2 \rangle
\]

\[
(\text{ETKF, VLKF, truth})
\]
IV. Model error and forecast bias

We consider performance over standard ETKF

\[ S = \frac{\mathcal{E}^E}{\mathcal{E}^V}, \]

and over using the “poor man’s” analysis of observations and climatology

\[ \hat{S}^E = \frac{\hat{\mathcal{E}}}{\mathcal{E}^E} , \quad \hat{S}^V = \frac{\hat{\mathcal{E}}}{\mathcal{E}^V}. \]

\[ \Delta t_{\text{obs}} = 24\text{hrs} \quad \Delta t_{\text{obs}} = 36\text{hrs} \quad \Delta t_{\text{obs}} = 48\text{hrs} \]
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\[ \Delta t_{\text{obs}} = 24\text{hrs} \quad \Delta t_{\text{obs}} = 36\text{hrs} \quad \Delta t_{\text{obs}} = 48\text{hrs} \]

**Trade-off:** The smaller \( \gamma \), the better skill over ETKF, but the less skill compared with “poor man’s” analysis.
We have here

- derived a variance limiting Kalman filter (VLKFilter) which adaptively damps unrealistic excitation of ensemble spread in underresolved regions
- applied this filter to a sparse observational grid
  - has better skill than ETKF for small ($\leq 6h$) observation times
  - has better skill for observables and pseudo-observables
- proposed a mechanism for blow-up filter divergence
- applied this filter to control balance
  - has better skill than DEnKF and controls unbalance
- applied this filter to model error (underdamping)
  - has better skill than ETKF for large observation intervals ($\geq 36h$)
  - trade-off between superior skill over ETKF and being better than observations/climatology

(GAG, Mitchell & Reich, MWR 2011; Mitchell & GAG, QJRMS 2012, in press)
Lorenz-96 model

How is the skill distributed over

observables

pseudo-observables

Constraining overestimation of error covariances in ensemble K

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Lorenz-96 model

Dependency on observational noise level $R_{\text{obs}} = (\eta \sigma_{\text{clim}})^2 \mathbf{I}$, $N_{\text{obs}} = 4$

$\Delta t_{\text{obs}} = 0.025$
(1 hour)

$\Delta t_{\text{obs}} = 0.05$
(2 hours)

$\Delta t_{\text{obs}} = 0.25$
(5 hours)

Red: Observations    Green: ETKF    Blue: VLK
Constraining the covariances produces more reliable ensembles – even in the case when there is no skill improvement with $S = 1$

**Ranked probability histograms for the observables**

- sort the forecast ensemble $X_f = [x_{f,1}, x_{f,2}, \ldots, x_{f,k}]$ and create bins $(-\infty, x_{f,1}], (x_{f,1}, x_{f,2}], \ldots, (x_{f,k}, \infty)$ at each forecast step
- increment whichever bin the actual truth falls into at each forecast step

Convex histogram: underestimating ensemble
Concave histogram: overestimating ensemble
Flat histogram: reliable ensemble for which each ensemble member has equal probability of being nearest to the truth

$$\gamma = 0.5 \text{ and } \Delta t_{\text{obs}} = 48\text{hrs}$$

![Graphs showing ranked probability histograms for different ensemble sizes](image)

$N_{\text{obs}} = 3 \quad N_{\text{obs}} = 4 \quad N_{\text{obs}} = 8$

(ETKF \quad VLKF)
Lorenz-96 model

There is an order of magnitude difference between the RMS errors for the observables and the pseudo-observables for large $N_{obs}$. This suggests that the information of the observed variables does not travel too far away from the observational sites.

Remark: The advective time scale of the Lorenz-96 system is much smaller than $\Delta t_{obs}$ which explains why the skill is not equally distributed over the sites, and why, especially for large values of $N_{obs}$ we observe a big difference between the site-averaged skills of the observed and unobserved variables.
Application: Model error and forecast bias

Constraining overestimation of error covariances in ensemble Kalman filters

$P_f$: $\gamma = 1 \quad \gamma = 0.9 \quad \gamma = 0.8$ ($\Delta t_{obs} = 48$ hrs) continuous: ETKF  dashed: VLKF

$P_a$: $\gamma = 0.9 \quad \gamma = 0.8$ ($\Delta t_{obs} = 48$ hrs) continuous: ETKF  dashed: VLKF
Application: Model error and forecast bias

\[ Pf: \]

\[ Pa: \]

\[ \gamma = 1 \quad \gamma = 0.9 \quad \gamma = 0.8 \quad (\Delta t_{\text{obs}} = 48\text{hrs}) \quad \text{continuous: ETKF \ dashed: VLKF} \]

- increased sparsity and model error lead to overestimation
- covariances of observables are also limited for VLKF
Distribution of RMS error over observables and pseudo-observables

\[ N_{\text{obs}} = 3 \quad N_{\text{obs}} = 4 \quad N_{\text{obs}} = 8 \]

\[(\text{ETKF/VLKF}) \quad (\Delta t_{\text{obs}} = 48 \text{ hours}) \]
Distribution of RMS error over observables and pseudo-observables

- VLKF outperforms ETKF with increasing model error $\gamma$ and increasing sparsity $N_{\text{obs}}$
- VLKF becomes less effective compared to poor man’s analysis with increasing model error $\gamma$ and increasing sparsity $N_{\text{obs}}$
- issue of overestimation

\[ N_{\text{obs}} = 3 \quad N_{\text{obs}} = 4 \quad N_{\text{obs}} = 8 \]

(ETKF/VLKF)

(\(\Delta t_{\text{obs}} = 48\) hours)
Distribution of RMS error over observables and pseudo-observables

- VLKF outperforms ETKF with increasing model error $\gamma$ and increasing sparsity $N_{\text{obs}}$
- VLKF becomes less effective compared to poor man’s analysis with increasing model error $\gamma$ and increasing sparsity $N_{\text{obs}}$
- issue of overestimation

- ETKF outperforms VLKF for decreasing model error and decreasing sparsity
- ETKF performs worse than poor man’s analysis for small model error

- convergence of ETKF and VLKF to $\mathcal{E}_o^* < \sqrt{R_{\text{obs}}}$ for sufficiently large model error
- issue of underestimation

$N_{\text{obs}} = 3 \quad N_{\text{obs}} = 4 \quad N_{\text{obs}} = 8$

(ETKF/ VLKF)

$(\Delta t_{\text{obs}} = 48 \text{ hours})$
Dependency of skill on observation noise $R_{obs} = (\eta \sigma_{clim})^2 I$.

VLKF outperforms ETKF and the poor man’s analysis for sufficiently large observational noise since large noise gives more preference to the pseudo-observables which are controlled by the VLKF.

- trade-off between skill over ETKF and efficacy over poor man’s analysis with increasing model error.

$\eta = 0.15 \quad \eta = 0.15 \quad \eta = 0.5 \quad \eta = 1$

$(\Delta t_{obs} = 48 \text{ hours}, \ N_{obs} = 4)$
Application: Model error and forecast bias

How is the RMS error distributed over the observables and the pseudo-observables ($\Delta t_{\text{obs}} = 48\text{hrs}$)?

Constraining overestimation of error covariances in ensemble K

Toulouse, November 13th 2012
Model Error

Dependency of skill on sparsity $N_{\text{obs}}$

![Graph showing the dependency of skill on sparsity $N_{\text{obs}}$ for ETKF and VLKF methods.](image)

Constraining overestimation of error covariances in ensemble $K$  
Toulouse, November 13th 2012
Dependence of climatic mean (left) and variance (right) on damping parameter $\gamma$. As well as the climatic variance (black, circles, solid) we show the average variance calculated over forecast intervals $\Delta t_{\text{obs}} = 12$ hours (red, squares, dotted), $\Delta t_{\text{obs}} = 24$ hours (green, triangles, dash-dotted) and $\Delta t_{\text{obs}} = 48$ hours (blue, crosses, dashed).
Overestimation of forecast error covariance

Finite size ensemble sizes can lead to

- underestimation of diagonal elements of the forecast error covariance $P_f$
- overestimation of off-diagonal elements of the forecast error covariance $P_f$

To control off-diagonal terms one uses *localization* \cite{Houtekamer1998}, for example

$$P_f \rightarrow C_{\text{loc}} \circ P_f$$

Can VLKF act as a form of localization?
Overestimation of forecast error covariance

Finite size ensemble sizes can lead to

- underestimation of diagonal elements of the forecast error covariance $P_f$
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To control off-diagonal terms one uses localization (Houtekamer and Mitchell (1998)), for example

$$P_f \rightarrow C_{loc} \circ P_f$$

Can VLKF act as a form of localization?

2-d example $z = (x, y)$ where only $x$ is observed:

$$\kappa_{loc} = \frac{1}{P_{f11} + R} \begin{pmatrix} P_{f11} & 0 \\ C_{loc21} & P_{f21} \end{pmatrix}$$

$$\kappa_{VLKF} = \kappa_{VLKF}(R_w):$$

- $R_w \rightarrow \infty$: $\kappa_{VLKF} \rightarrow \frac{1}{P_{f11} + R} \begin{pmatrix} P_{f11} & 0 \\ P_{f21} & 0 \end{pmatrix}$

- $R_w \rightarrow \varepsilon R_w$, $R \rightarrow \frac{1}{\varepsilon} R$: $\kappa_{VLKF} \rightarrow \begin{pmatrix} \varepsilon \frac{P_{f12}}{P_{f22}} R & 0 \\ \varepsilon^2 \frac{P_{f12}}{P_{f22}} R & \frac{P_{f12}}{P_{f22}} \end{pmatrix}$ with $\vartheta = \det(P_f)$