A possible implementation of the 4D-Var based on a 4D-ensemble

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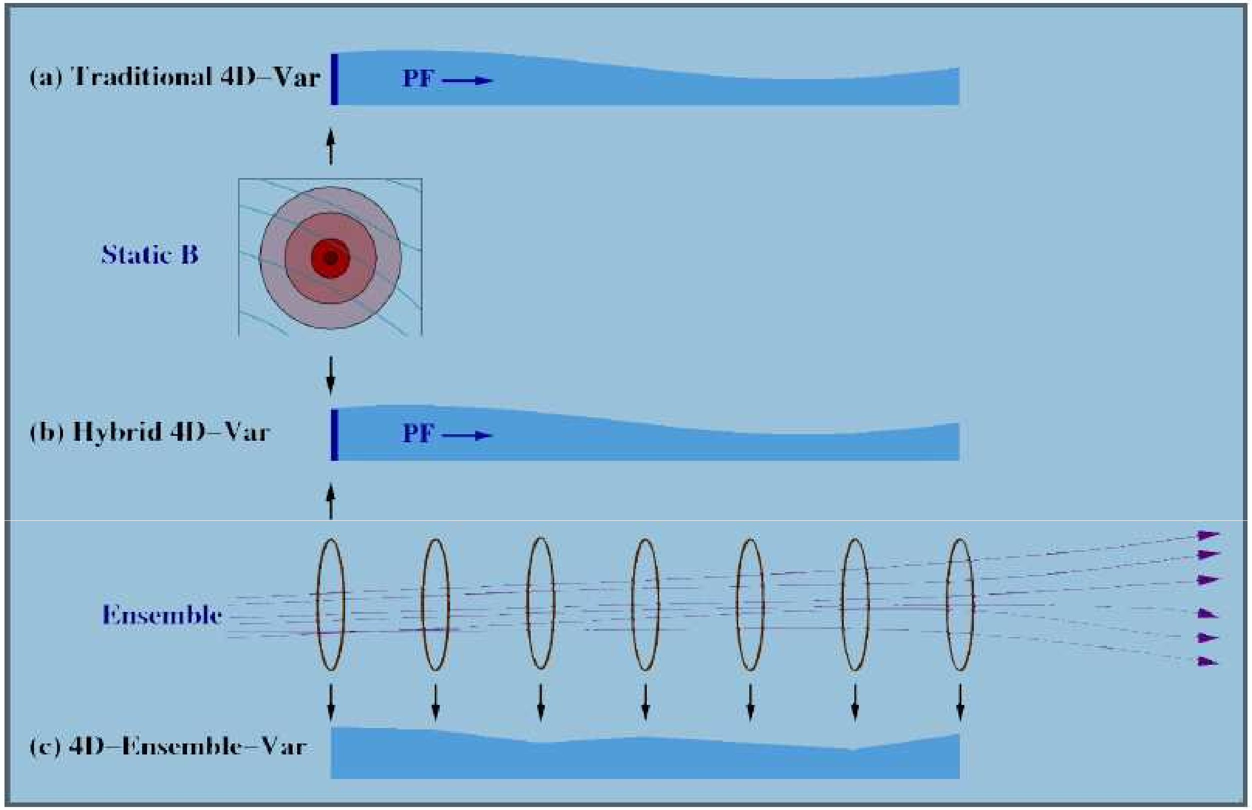
Introduction

- 4D-Var
- ✓ Simplified description of **B** at initial time.
- ✓ Possible improv. via an ens. of pert. 4D-Var (Météo-France, ECMWF) : spat./temp. variations of error variances and correlations (wavelets).
- ✓ Difficult development and maintenance of TL/AD.
- ✓ Poor scalability of TL/AD
- ✓ Linear evolution of covariances.

- 4D-Var based on a 4D ensemble : 4D-En-Var
- ✓ Similar to EnKF.
- ✓ Keeps benefits of 4D-Var (global analysis, add. terms, outer-loop, …)
- ✓ Localization of the raw covariances made in model space.
- ✓ Minimization cost similar to 3D-Var.
- ✓ Natural parallelization.
- ✓ Covariances given by NL forecasts.

4D-Var / Hybrid 4D-Var / 4D-En-Var



Barker and Clayton, 2011

4D-En-Var formulation

- Minimization of

$$J(\underline{\delta \mathbf{x}}) = \underline{\delta \mathbf{x}}^T \underline{\mathbf{B}}^{-1} \underline{\delta \mathbf{x}} + (\underline{\mathbf{d}} - \underline{\mathbf{H}}\ \underline{\delta \mathbf{x}})^T \underline{\mathbf{R}}^{-1} (\underline{\mathbf{d}} - \underline{\mathbf{H}}\ \underline{\delta \mathbf{x}}), \text{ with}$$

$\underline{\mathbf{d}}$ 4D vector of the innovations distributed in time,
$\underline{\mathbf{H}}$ 4D linearized observation operator,
$\underline{\mathbf{R}}$ 4D (but diagonal!) covariance matrix of obs. errors,
$\underline{\delta \mathbf{x}}$ 4D vector of the increments to be added to the 4D bg $\mathbf{x}^b$,
$\underline{\mathbf{B}}$ 4D covariance matrix of bg errors, given by an ensemble.

(Lorenc, 2012)

4D-En-Var formulation

$$\underline{\mathbf{X}}^{f} = (\underline{\mathbf{x}}^{f'}{}_{1}, \ldots, \underline{\mathbf{x}}^{f'}{}_{L}),$$

where L is the ensemble size and

$$\underline{\mathbf{x}}^{f'}{}_{\ell} = \underline{\mathbf{x}}^{f}{}_{\ell} - \langle\underline{\mathbf{x}}^{f}\rangle \,/\, (L-1)^{1/2}, \ \ell=1,L,$$

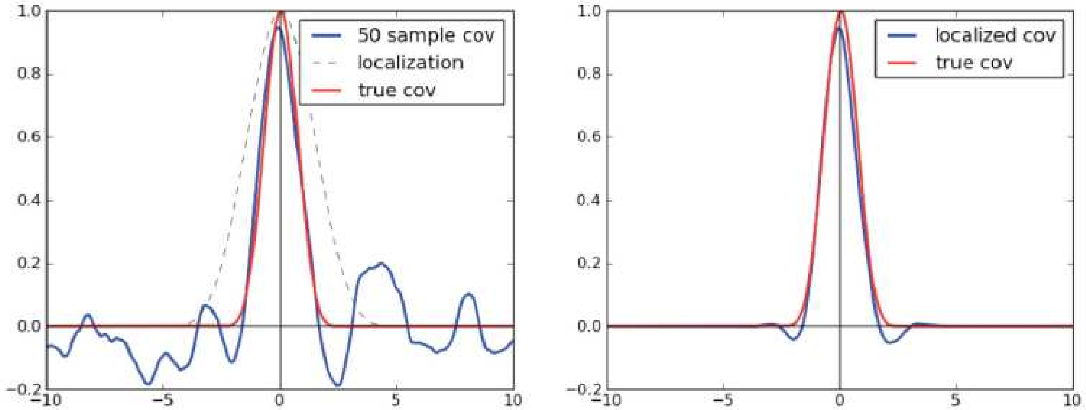
are the deviations of the 4D pert. forecasts from the mean ens. traject.

$$\underline{\mathbf{P}} = \underline{\mathbf{X}}^{f} (\underline{\mathbf{X}}^{f})^{T}$$

$$= \begin{pmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \ldots & \mathbf{P}_{0,K-1} \\ & \mathbf{P}_{1,1} & & \\ & & & \\ \mathbf{P}_{K-1,0} & \ldots & & \mathbf{P}_{K-1,K-1} \end{pmatrix}$$

where K is the number of times in the assimilation period.

Localization of the covariances



Whitaker, 2011

$$\underline{B} = \underline{P} \circ \underline{C}$$

$$= \begin{pmatrix} P_{0,1} & P_{0,1} & \dots & P_{0,K-1} \\ & P_{1,1} & & \\ & & & \\ P_{K-1,0} & \dots & & P_{K-1,K-1} \end{pmatrix} \circ \begin{pmatrix} C_{0,1} & C_{0,1} & \dots & C_{0,K-1} \\ & C_{1,1} & & \\ & & & \\ C_{K-1,0} & \dots & & C_{K-1,K-1} \end{pmatrix}$$

Implementations of 4D-En-Var

- $\delta\mathbf{x} = \mathbf{B}^{1/2}\chi$
  $= (\mathbf{P} \circ \boldsymbol{C})^{1/2}\chi$
  $= \Sigma_{\ell=1,L}\, \mathbf{x}^{f'}_{\ell} \circ (\boldsymbol{C}^{1/2}\chi_{\ell})$

  $J^b(\chi) = \Sigma_{\ell=1,L}\, \chi_{\ell}^T\chi_{\ell}$, dim $\chi = N(N_c)xL$ (or KxN_cxL in 4D with model error)

  Use of a Conjugate Gradient (CG) with $\mathbf{B}^{1/2}$ change of variables.

  (Buehner 2005, 2010)

- $\delta\mathbf{x} = \Sigma_{\ell=1,L}\, \mathbf{x}^{f'}_{\ell} \circ \alpha_{\ell}$, with $\alpha_{\ell} = \boldsymbol{C}^{1/2}\chi_{\ell}$

  $J^b(\alpha) = \Sigma_{\ell=1,L}\, \alpha_{\ell}^T\boldsymbol{C}^{-1}\alpha_{\ell}$, dim $\alpha = N(N_c)xL$ (or KxN_cxL in 4D with mod. error)

  Use of a Double Preconditioned CG (DPCG) with $\boldsymbol{C}$ preconditioning.

  (Lorenc, 2003; Wang et al 2007; Wang 2010)

Possible alternative impl. of 4D-En-Var: 4D $\underline{\mathbf{B}}$ preconditioning

Use of DPCG, with 4D covariance matrix $\underline{\mathbf{B}}$ and Hessian J" = $\underline{\mathbf{B}}^{-1} + \underline{\mathbf{H}}^T \underline{\mathbf{R}}^{-1} \underline{\mathbf{H}}$.
In this case, the dimension of the control variable $\delta\underline{\mathbf{x}}$ is KxN.

$\delta\underline{\mathbf{x}}_0 = 0$
$\underline{\mathbf{g}}_0 = - \underline{\mathbf{H}}^T \underline{\mathbf{R}}^{-1} \underline{\mathbf{d}}$
$\underline{\mathbf{h}}_0 = \underline{\mathbf{B}}\, \underline{\mathbf{g}}_0$
$\underline{\mathbf{d}}_{-1} = \underline{\mathbf{e}}_{-1} = 0$
loop over n

$\underline{\mathbf{d}}_n = -\underline{\mathbf{h}}_n + \beta_{n-1} \underline{\mathbf{d}}_{n-1}$
$\underline{\mathbf{e}}_n = -\underline{\mathbf{g}}_n + \beta_{n-1} \underline{\mathbf{e}}_{n-1}$
$\underline{\mathbf{f}}_n = \underline{\mathbf{e}}_n + \underline{\mathbf{H}}^T \underline{\mathbf{R}}^{-1} \underline{\mathbf{H}}\, \underline{\mathbf{d}}_n$
$\alpha_n = \underline{\mathbf{g}}_n^T \underline{\mathbf{h}}_n / \underline{\mathbf{d}}_n^T \underline{\mathbf{f}}_n$
$\underline{\mathbf{g}}_{n+1} = \underline{\mathbf{g}}_n + \alpha_n \underline{\mathbf{f}}_n$
$\underline{\mathbf{h}}_{n+1} = \underline{\mathbf{B}}\, \underline{\mathbf{g}}_{n+1}$
$\delta\underline{\mathbf{x}}_{n+1} = \delta\underline{\mathbf{x}}_n + \alpha_n \underline{\mathbf{d}}_n$
$\beta_n = \underline{\mathbf{g}}_{n+1}^T \underline{\mathbf{h}}_{n+1} / \underline{\mathbf{g}}_n^T \underline{\mathbf{h}}_n$

(Derber and Rosati, 1989; El Akkraoui et al, 2012)

Multiplication by $\underline{\mathbf{B}}$

- Application of DPCG : $\underline{\mathbf{h}} = \underline{\mathbf{B}}\,\underline{\mathbf{g}}$

- $\underline{\mathbf{B}} = \underline{\mathbf{P}} \circ \underline{\mathbf{C}}$

  $\mathbf{C} = \mathbf{S}^{-1}\,\mathbf{C}^{s}\,\mathbf{S}$

- $\mathbf{B}_{k,k'}\,\mathbf{g} = (\,\mathbf{X}^{f'}{}_{k}\,(\,\mathbf{X}^{f'}{}_{k'}\,)^{T} \circ \mathbf{C}\,)\,\mathbf{g}$

  $= \Sigma_{\ell=1,L}\,\mathbf{x}^{f'}{}_{k,\ell} \circ S^{-1}(\,\mathbf{C}^{s}\;S\;(\;\mathbf{x}^{f'}{}_{k',\ell} \circ \mathbf{g}\;)\,),$

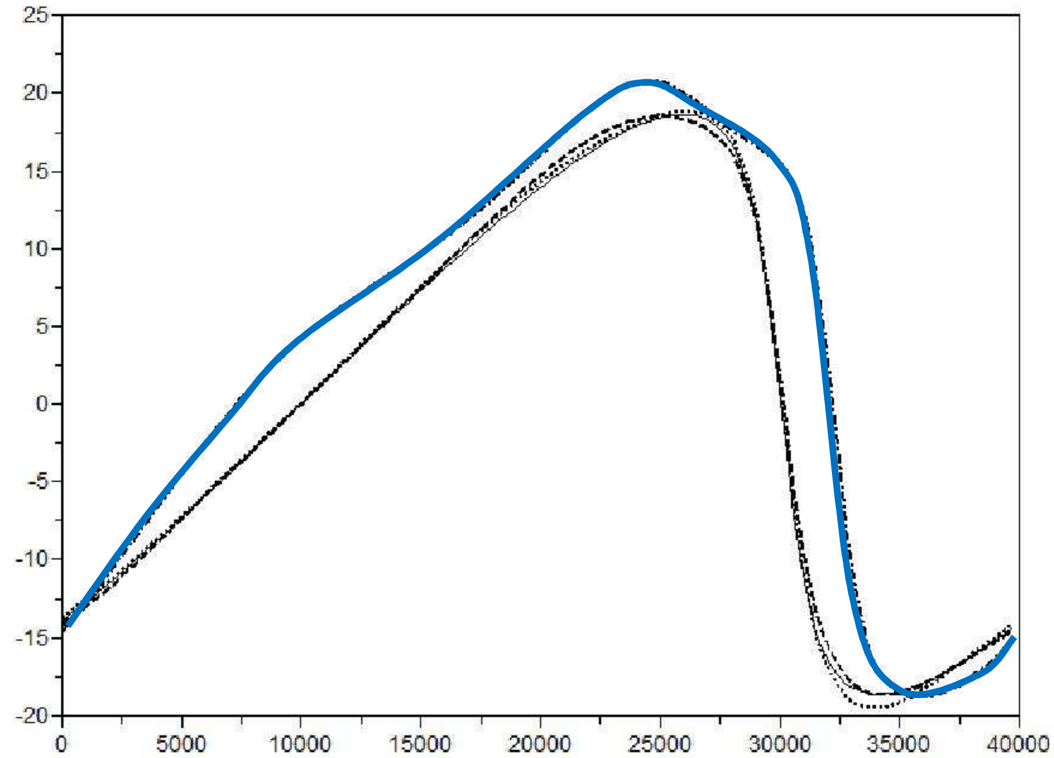
  where k and k' are two time indexes, and omitting the vertical dimension.

- Compact and meaningful expression of blocks of $\underline{\mathbf{B}}$.

- No need of extra variables (α_ℓ of χ_ℓ , with $\alpha_\ell = \mathbf{C}^{1/2}\,\chi_\ell$).

- Can be parallelized over K (time), N (space), L (ensemble), …

Application to the Burgers model



Forecasts from
- reference (black solid)
- $\mathbf{x}^b$ (blue dashed-dotted)
- $\mathbf{x}^a$ 4D-En-Var (black dashed)
- $\mathbf{x}^a$ 4D-Var (black dotted)

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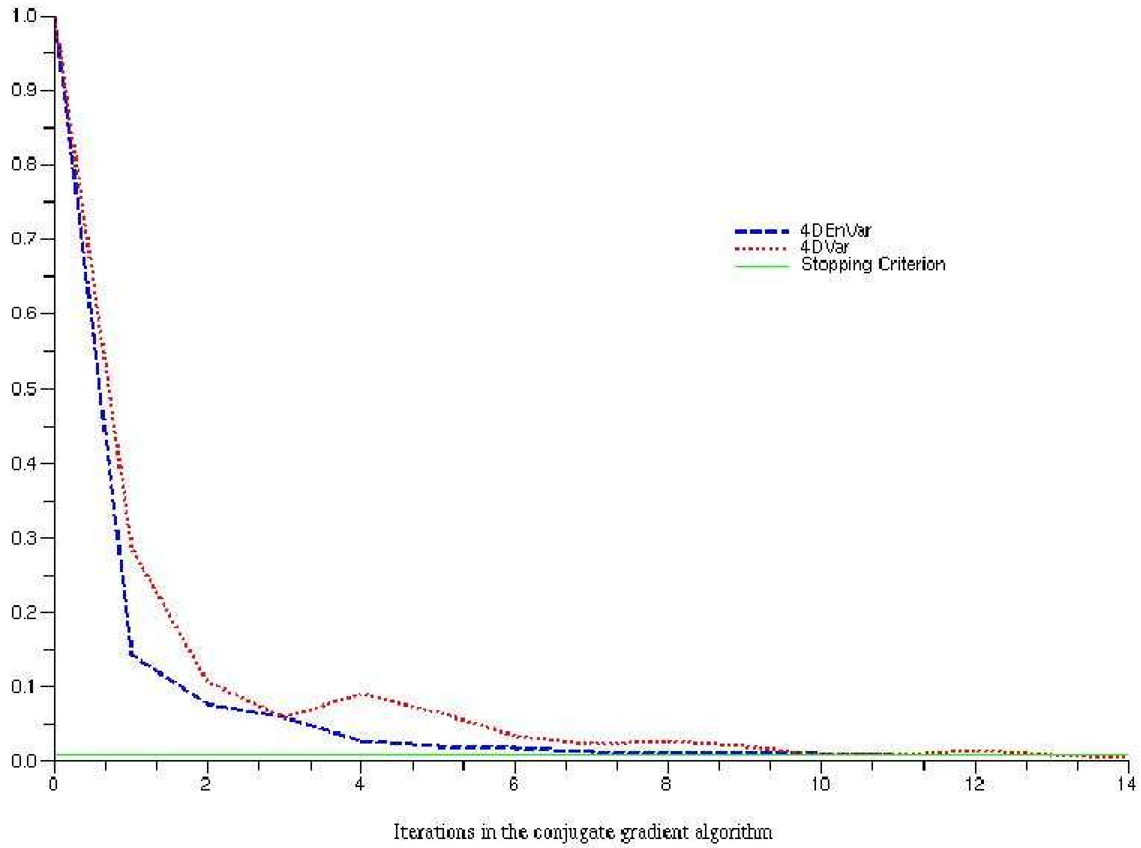
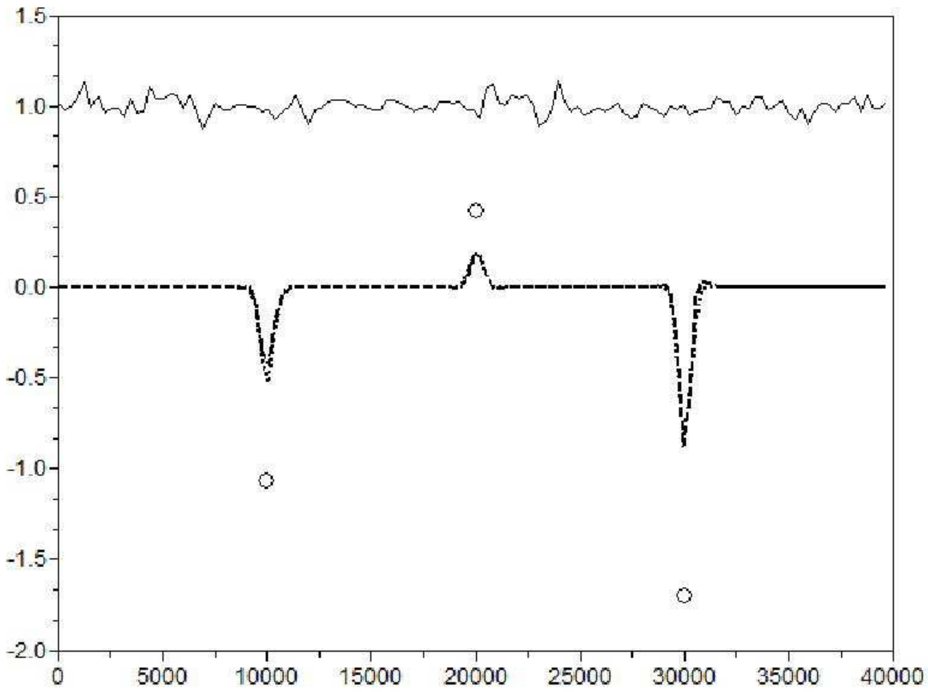
Convergence
4D-En-Var / 4D-Var

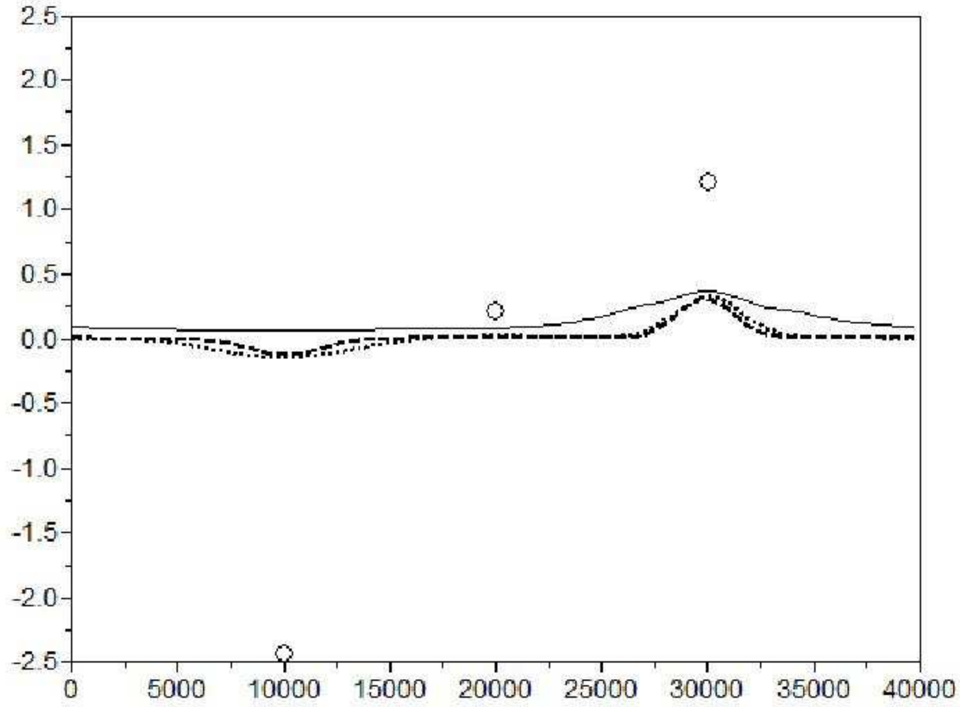


Decrease of the gradient norm

- 4D-En-Var (blue dashed)
- 4D-Var (red dotted)

Increment comparison
4D-En-Var / 4D-Var

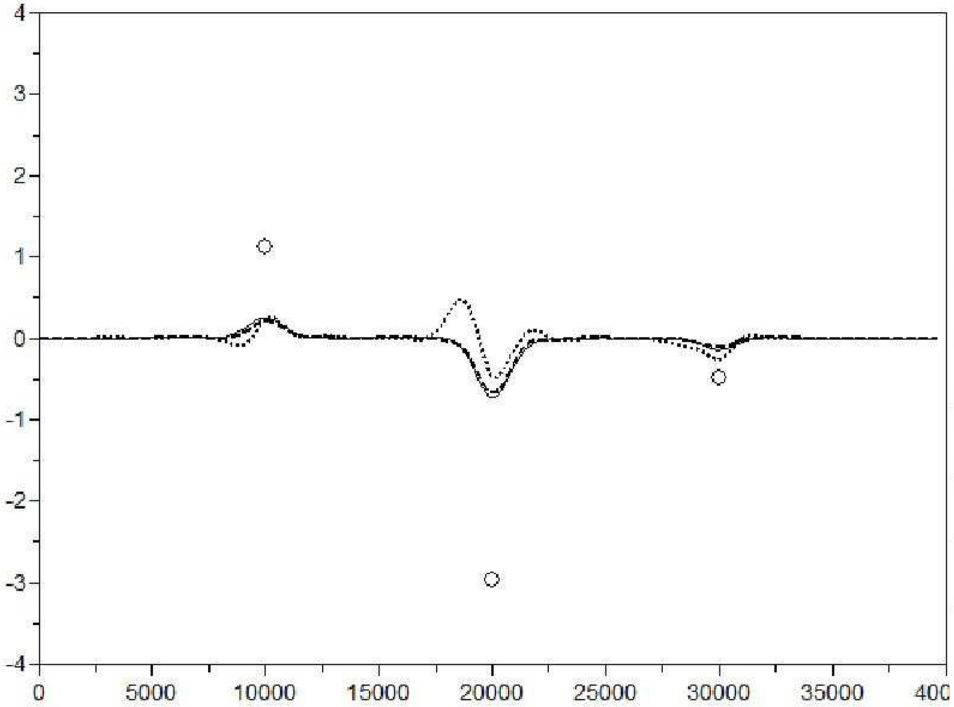


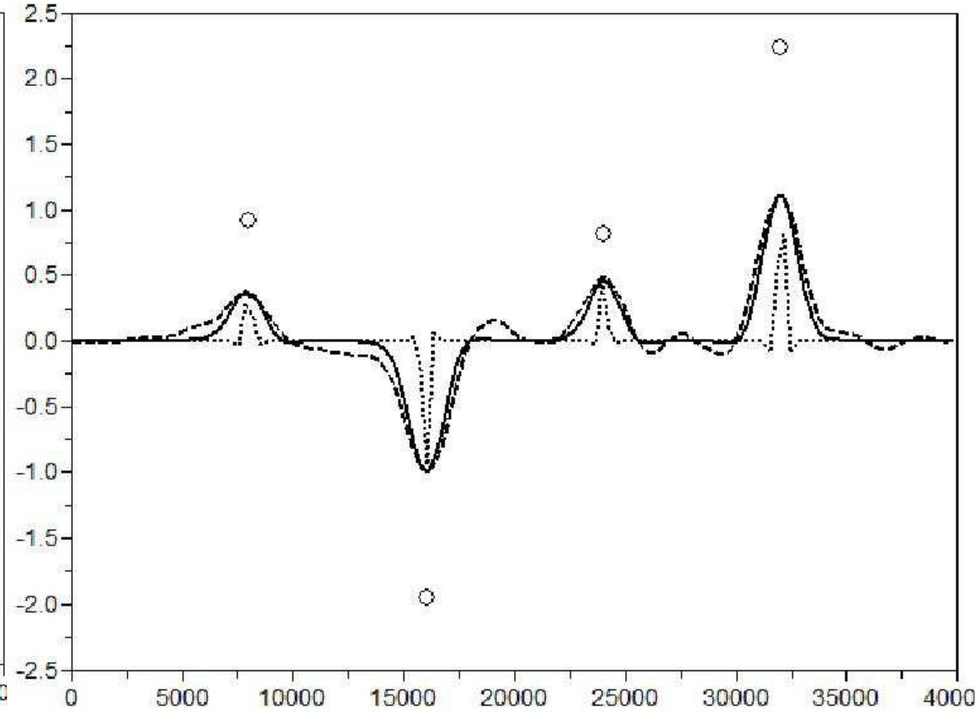


Observations at t_0
$\delta \mathbf{x}_0$ at t_0 :
- 4D-En-Var (dashed)
- 4D-Var (dotted)
- bg error square-root (solid)

Observations at t_f (t_0+6h)
$\delta \mathbf{x}_f$ at t_f :
- 4D-En-Var (dashed)
- 4D-Var (dotted)
- bg error square-root (solid)

4D-En-Var parameters





Ensemble size:
- L = 5 (dotted)
- L = 100 (dashed)
- L = 1000 (solid)

Localization length scale:
- LC = 50 km (dotted)
- LC = 1200 km (solid)
- LC = 5000 km (dashed)

Conclusion

- 4D-En-Var : combination of 4D-Var and EnKF.
- Treats all observations simultaneously.
- Localization of covariances made in model space.
- Natural parallelization over time, space and ensemble.
- Possible representation of model error inside the assimilation window.
- Efficient preconditioning available (as in strong constraint 4D-Var).
- DPCG with 4D ens. **B** matrix : possible implementation of 4D-En-Var.