Bayesianity of Ensemble Variational Assimilation

Mohamed Jardak and Olivier Talagrand

Laboratoire de Météorologie Dynamique
École Normale Supérieure
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- **Objectives**
  1. Objectively evaluate the Ens/4D-Var as an ensemble estimator in the non-linear and non-Gaussian cases.
  2. Evaluate as far as possible the Bayesianity of the ensemble produced in the non-linear and non-Gaussian cases .
  3. Compare with other existent ensemble algorithm schemes (EnKF and PF) .

Bayesianity

Under linearity and gaussianity, the following algorithm achieves Bayesian estimation

- Given the data

$$z = \Gamma x + \zeta, \quad \zeta \in \mathcal{N}([\mu, \Sigma])$$

- The conditional posterior probability distribution is

$$P(x|z) = \mathcal{N}([x^a, P^a])$$

with

$$x^a = \left(\Gamma^T \Sigma^{-1} \Gamma\right)^{-1} \Gamma^T \Sigma^{-1}(z - \mu) \text{ and } P^a = \left(\Gamma^T \Sigma^{-1} \Gamma\right)^{-1}$$

Ready recipe for producing sample of independent realizations of posterior probability distribution :

Perturb data vector additively according to error probability distribution $\mathcal{N}([\mu, \Sigma])$, and compute analysis x^a for each perturbed data vector.

Variational assimilation

The following algorithm produces a sample of independent realizations of the probability distribution of the state of the system, conditioned by the data x_0^b and y_k.

- Available data
  1. Background estimate at $t = 0$, $x_0^b = x_0 + \xi_0^b$, $\xi_0^b \in \mathcal{N}([0, \mathrm{B}^{1/2}])$
  2. Observations at $t = 0$, $Yy_k = \mathrm{H}_k x_k + \epsilon_k, \epsilon_k \in \mathcal{N}([0, [\mathrm{R}_k]^{1/2}])$
  3. Model (supposed to be exact) $x_{k+1} = \mathrm{M}_k x_k$, and $k = 0, \cdots, K-1$
  4. Errors ξ_0^b and ϵ_k assumed to be unbiased and uncorrelated in time.
  5. H_k and M_k assumed linear
- The optimal state (mean of the Bayesian Gaussian pdf) at $t = 0$ minimizes the objective function

$$\begin{cases} \mathfrak{J}(\xi_0) = \frac{1}{2}(x_0^b - \xi_0)^T [\mathrm{B}]^{-1} (x_0^b - \xi_0) + \frac{1}{2}\sum_k (y_k - \mathrm{H}_k\xi_k)^T \mathrm{R}_k^{-1} (y_k - \mathrm{H}_k\xi_k) \\ \xi_{k+1} = \mathrm{M}_k \xi_k \end{cases}$$

What happens under nonlinearity and non-Gaussianity ?

Ens/4D-Var method

- for iens $= 1$: Nens

  1. perturb the data
     - $x_0^b(iens) \in \mathcal{N}(\bar{x}_0^b, \mathrm{B}^{1/2})$
     - $y_k(iens) \in \mathcal{N}(\bar{y}_k, \mathrm{R}_k^{1/2})$
  2. perform a 4D-Var to find the optimal initial ensemble member solution .

$$x_0^{opt}(iens) = \min_{X \in \mathfrak{A}} \mathfrak{J}_{iens}(X)$$

  3. find the optimal ensemble member trajectory.

$$x^{opt}(t, iens) = \underbrace{\mathfrak{M}_{t \to 0}}_{non-linear\ \ model} (x_0^{opt}(iens))$$

- end for

Results

How to objectively evaluate the Bayesian character of an ensemble estimation procedure ?

- Bayesianity implies reliability therefore lack of reliability implies lack of Bayesianity.

reliability is the statistical agreement between the predicted probability of occurrence and the observed frequency of occurrence.

- Consistency : Under linearity the expectation of the objective function at its minimum is half the number of observations p

$$\mathbb{E}(\mathfrak{J}(x_{opt})) = \frac{p}{2}$$

and under Gaussianity we have

$$Var(\mathfrak{J}(x_{opt})) = p.$$

Testing reliability

- rank histogram.
- reliability diagram.
- Brier scores :

$$\mathbb{B} = \underbrace{\frac{1}{p_c(1-p_c)}}_{uncertainty} \left[\underbrace{\int_0^1 (p' - p)^2 g(p) dp}_{\mathbb{B}_{sc}} + \underbrace{\int_0^1 p'(1-p') g(p) dp}_{\mathbb{B}_{sv}} \right]$$

  - p predicted probability.
  - g the frequency with which p has been predicted.
  - $p'(p)$ observed frequency.
  - p_c the frequency of occurence of the event $\mathscr{E}$ under observation.

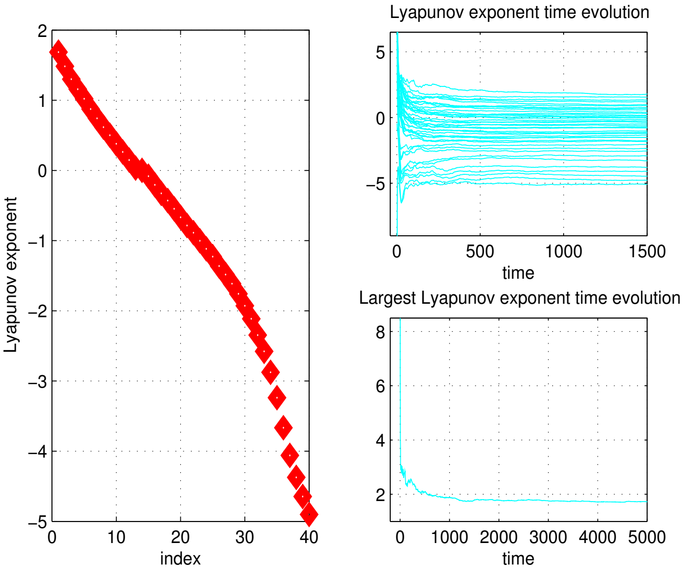
The Lorenz96 model

- Forward model

$$\frac{dX_k}{dt} = (X_{k+1} - X_{k-2})X_{k-1} - X_k + F \quad \text{for} \quad k = 1, \cdots, N$$

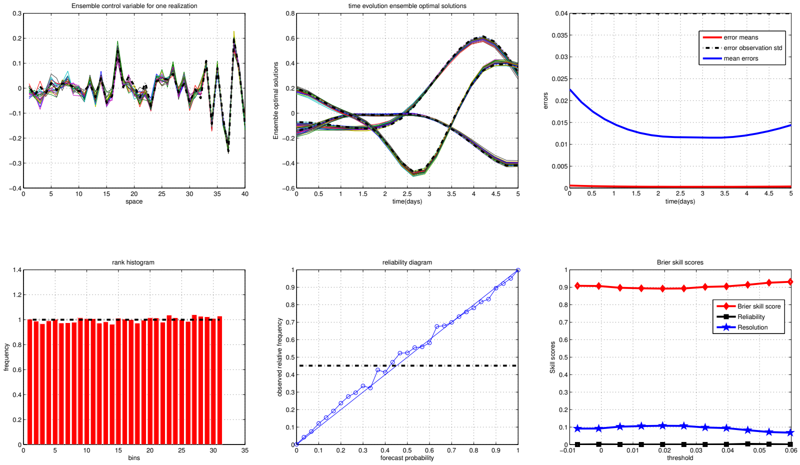
- Set-up parameters :
  1. the index k is cyclic so that $X_{k-N} = X_{k+N} = X_k$.
  2. $F = 8$, external driving force.
  3. X_k, a damping term.
  4. $N = 40$, the system size.
  5. $Nens = 30$, number of ensemble members.
  6. $\frac{1}{\lambda_{max}} \simeq 2.5 days$, λ_{max} the largest Lyapunov exponent.
  7. $\Delta t = 0.05 = 6 hours$, the time step.

Lyapunov exponents :



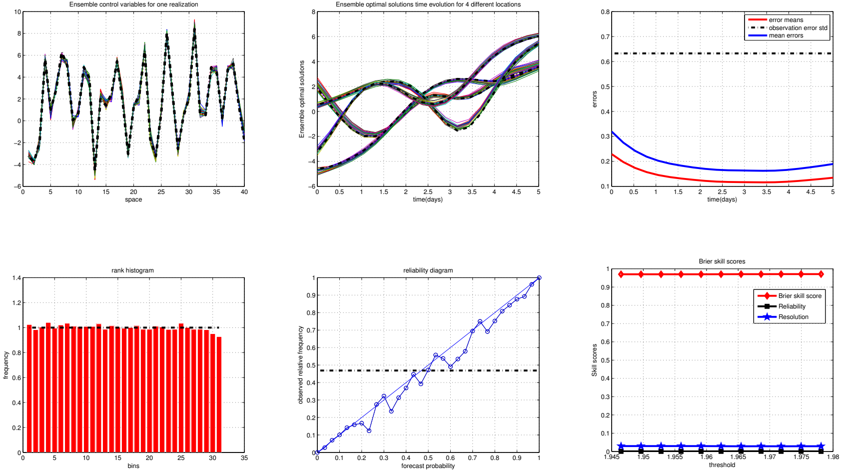
Results : The Lorenz96 model

Linear case : 5 days time length



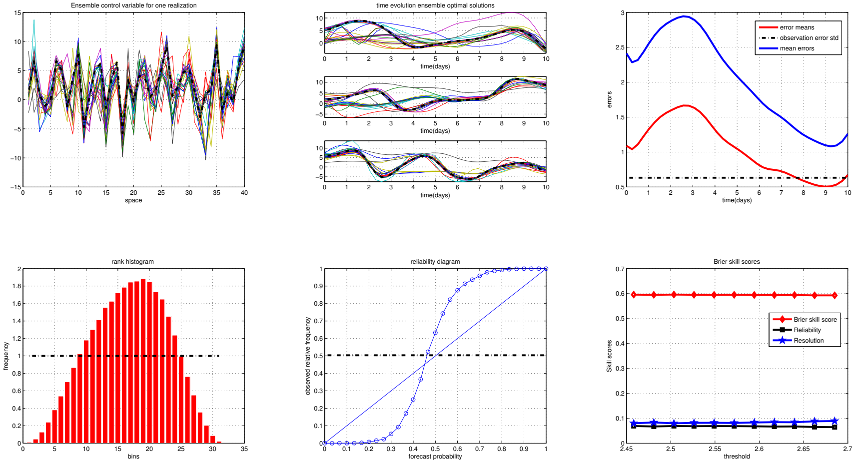
Results : The Lorenz96 model

Nonlinear case : 5 days time length

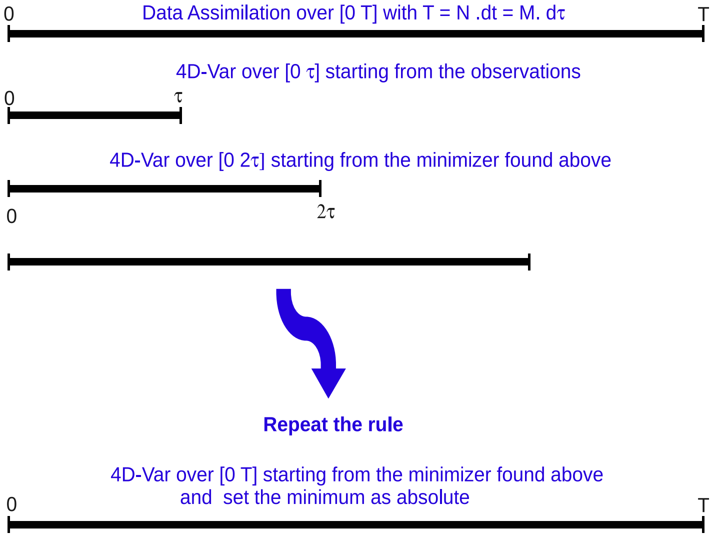


Results : The Lorenz96 model

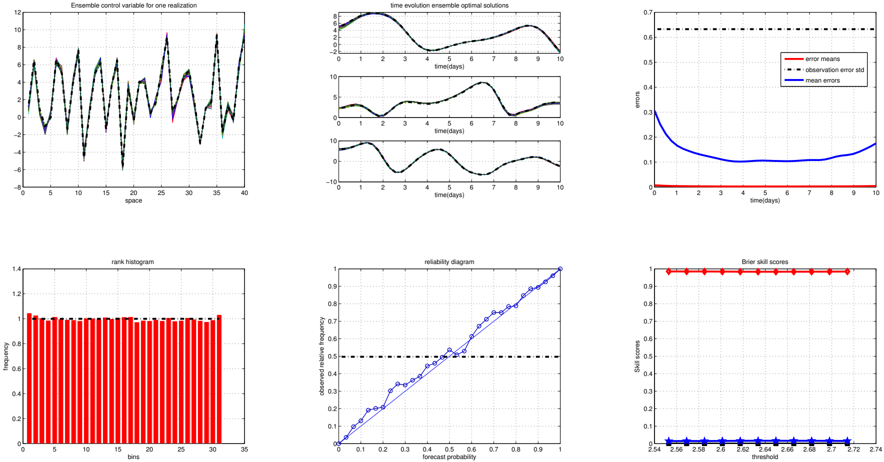
Nonlinear case : 10 days time length



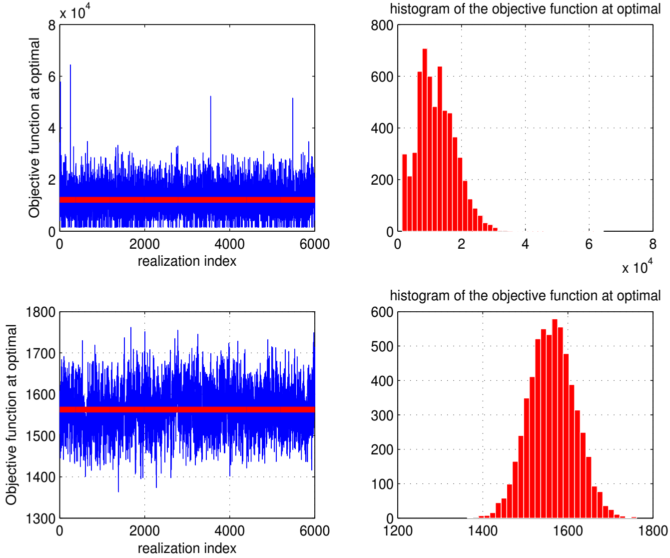
Quasi-Static Variational Assimilation (QSVA)



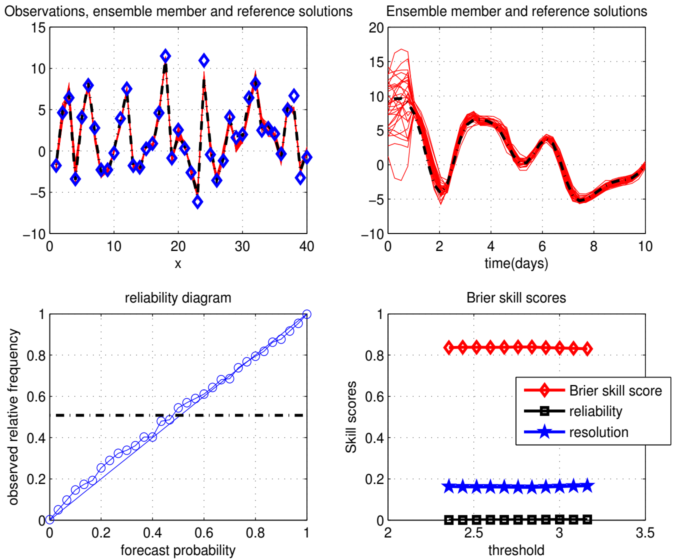
Nonlinear case : 10 days time length with QSVA



Consistency :

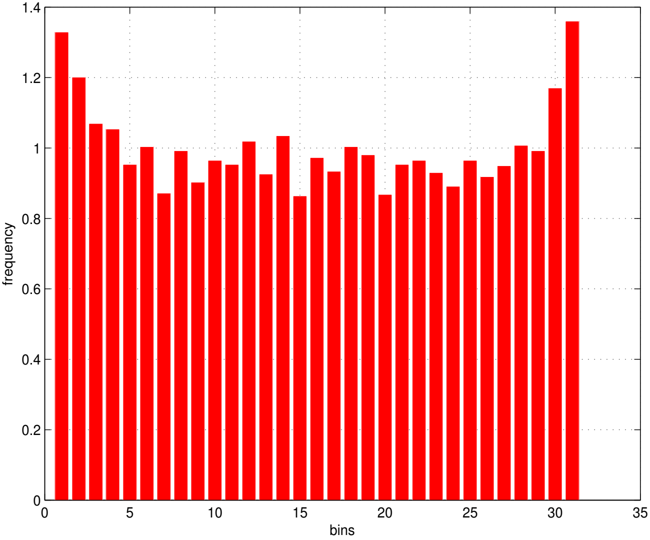


EnKF :

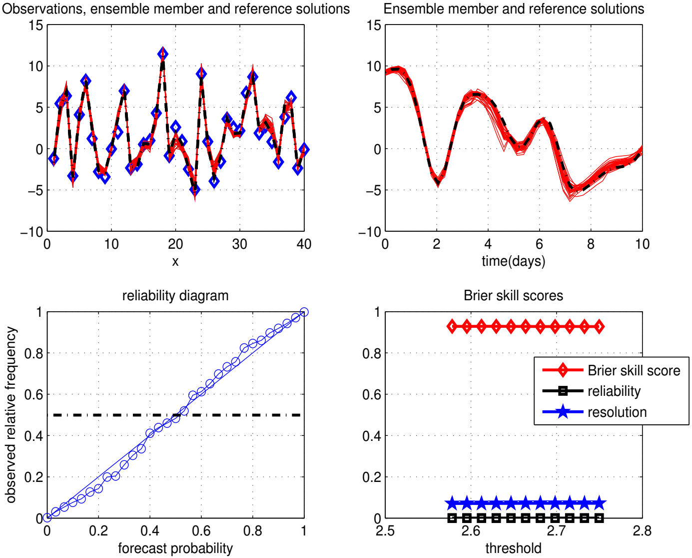


Observations, ensemble member and reference solutions
x
Ensemble member and reference solutions
time(days)
reliability diagram
observed relative frequency
forecast probability
Brier skill scores
Skill scores
threshold
Brier skill score
reliability
resolution

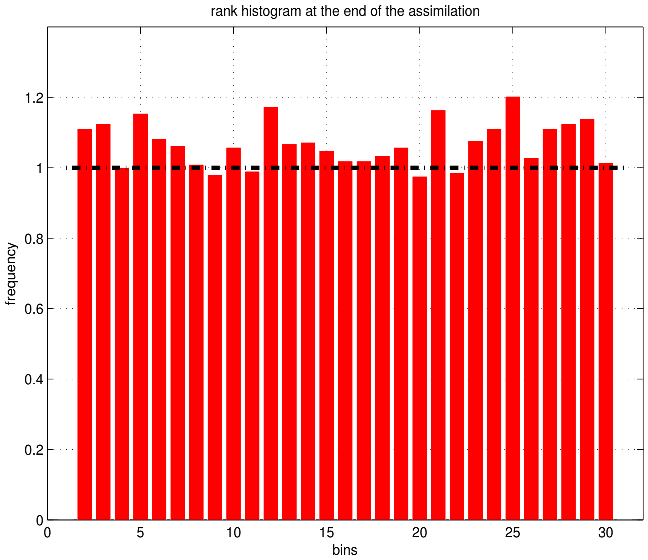
EnKF : rank histogram



PF :


Observations, ensemble member and reference solutions
15
10
5
0
−5
−10
0
10
20
30
40
x
Ensemble member and reference solutions
15
10
5
0
−5
−10
0
2
4
6
8
10
time(days)
reliability diagram
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
observed relative frequency
forecast probability
Brier skill scores
1
0.8
0.6
0.4
0.2
0
2.5
2.6
2.7
2.8
threshold
Brier skill score
reliability
resolution

PF : rank histogram



Summary

- Under non-linearity and non-Gaussianity the Ens/4D-Var is a reliable and consistent ensemble estimator (provided the QSVA is used for long DA windows) .
- Ens/4D-Var is at least as good an estimator as EnKF and PF.
- Similar results have been obtained for the Kuramoto-Sivashinsky model.

Ens/4D-Var : Pros and cons

Pros

- Easy to implement when having a 4D-Var code
- Highly parallelizable
- No problems with algorithm stability (i.e. no ensemble collapse, no need for localization and inflation, no need for weight resampling)
- Propagates information in both ways and takes into account temporally correlated errors

Cons

- Costly (Nens 4D-Var assimilations)
- Empirical

Future work

- Evaluate the performance of Ens/4D-Var on a simple geophysical and meteorological model (Shallow water on the sphere model) (in progress).
- Investigate the 4D-VarAUS method.

Thank you