Matched Filter CNR, Diversity and Signal Detectivity for Deterministic and Random Coherent Ladar Signals

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Introduction

- We develop theoretical expressions for coherent ladar signal:
  - Wideband and matched filter CNR
  - Speckle coherence time
  - Intrinsic diversity (a measure of speckle noise mitigation, $M = \text{Speckle SNR}$)
  - Spectrum models valid for arbitrary integration times
    - long or short relative to the coherence time
  - Relationship between spectrum height and matched filter CNR (CNR$_n$) is developed
  - Relationship to signal detectivity

- Theoretical predictions are supported by Monte-Carlo simulation experiment results
Coherent Ladar CNR

\[ CNR_w = \frac{\eta_r Pr}{h \nu B} = \frac{\eta_r Pr T}{h \nu B T} = \frac{\eta_r E_r}{h \nu B T} = \frac{m_r}{BT} \]

**For signal integration over times short compared to the speckle coherence time**

\[ B = \frac{1}{T} \text{ and } CNR_n = m_r \]
Coherent Ladar CNR

- **For arbitrary integration times, T**

  \[
  CNR_w = \frac{\eta_r Pr}{h v B} = \frac{m_r}{BT}
  \]

- **Optimal matched filter Bandwidth related to characteristic speckle time**

  \[
  B = \frac{1}{\tau_s} \quad \text{and} \quad CNR_n = \frac{\tau_s m_r}{T} = \frac{m_r}{M}
  \]

  for \( T > \tau_s \)

  \[
  = m_{rs}
  \]

What are detailed descriptions of \( \tau_s \) and \( M \)?
Long Duration \((T >> \tau_s)\) Signal Diversity and Speckle Coherence Time

- Diversity is defined as the speckle limited signal SNR or reciprocal normalized power variance

\[
M \equiv \frac{E[P_s]}{\text{var}[P_s]} = \text{SNR}_{\text{speckle}}
\]

- There are many definitions for coherence time.
  - For very long integration times, \(T\), the definition that leads to the diversity, \(M\), converging to \(M = T/\tau_s\) is what we call the “speckle coherence time”

\[
M \approx T / \tau_s \quad \text{for} \quad T >> \tau_s
\]

- Goodman shows, Ch 6 Statistical Optics, that this coherence time is given by

\[
\tau_s = \int_{-\infty}^{\infty} |\Gamma(\tau)|^2 d\tau / \Gamma(0)^2 = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau
\]

  - This is a measure of the width of the normalized signal autocorrelation function
  - Parseval’s Theorem leads to an expression in terms of the signal spectrum (not shown)

- For a Gaussian autocorrelation function

\[
\tau_s = \sqrt{\pi / 2\tau_C} = 1.25\tau_C
\]

  - Where \(\tau_C\) is the exp[-1] coherence time
Matched Filter CNR and Integrated Speckle Diversity for Arbitrary Duration Signals

- For arbitrary duration signals, the matched filter CNR can be generalized
  - Coherent photoelectrons contribute to CNRn
  - Incoherent photoelectrons contribute to diversity
  - Total number of electrons = CNRn and diversity product

\[
CNR_n = \frac{m_r}{M}
\]
\[
m_r = M \cdot CNR_n
\]

For arbitrary duration signals, a general expression for the diversity of the temporally integrated coherent signal power is given by [Goodman Ch. 6]

\[
M^{-1} = \frac{1}{T} \int_{-\infty}^{\infty} \Lambda(t/T) |\gamma(\tau)|^2 d\tau
\]

- \(\Lambda(t)\) is the triangle function
- \(\gamma(t)\) is the normalized autocorrelation function

- For long integration times, the triangle function can be approximated as 1 and the integral converges to the speckle coherence time (see previous chart), leading to

\[
M \rightarrow T/\tau_s; \text{ for } T \gg \tau_s
\]

- For a Gaussian signal spectrum, this becomes

\[
M^{-1} = \frac{\text{erf}(\sqrt{\pi} \alpha)}{\alpha} - \frac{1}{\pi \alpha^2} \left(1 - \exp(-\pi \alpha^2)\right)
\]

- where \(\alpha = T/\tau_s\)
- For small \(\alpha\), \(M \Rightarrow 1\)
- For large \(\alpha\), \(M \Rightarrow \alpha = T/\tau_s\)
- when \(\alpha = 1\), \(M = 1.4636\)

Integrated Irradiance Diversity

Log CNR

Normalized Time (\(\alpha = T/\tau_s\)) 10.0

Diversity (M)
Spectral Questions to Consider

❖ What is the spectral model as a function of the integration time relative to the coherence time.

❖ How is the matched filter CNR related to the ratio of the spectral peak to the noise floor?
  ❖ Coherent CW signal: $\text{CNR}_n$
  ❖ Coherent Gaussian pulse: $\sqrt{2} \text{CNR}_n$
  ❖ Long duration random signal with Gaussian autocorrelation/spectrum: $\sqrt{2} \text{CNR}_n$

![Signal Periodogram](image)

- $T = 10 \tau_s$
- $\text{CNR}_n \sim 10$
- $M \sim 10$
- $\sim \sqrt{2} \text{CNR}_n$
Fully Coherent CW Signal: Short Duration \( (T << \tau_s) \) Signal Spectral Model

- FFT is a matched filter for a rectangular windowed sine wave
- Peak of FFT output should be \( E_s/N_{oe} = \text{CNR}_n \)
  - Because it is the matched filter

- Signal Spectrum is a sinc function

\[
S_s(f) = \text{CNR}_n \cdot \text{sinc}^2((f - f_c)T)
\]

\[
\text{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}
\]

- So with noise the final model is

\[
S_{sn}(f) = \left(\frac{N_0}{2}\right)(1 + \text{CNR}_n) \cdot \text{sinc}^2((f - f_c)T)
\]

- The peak spectral height above the unit normalized NSD is precisely equal to the matched filter CNR, \( \text{CNR}_n \)
Arbitrary Duration Spectral Model

◆ An arbitrary duration signal is the product of a rectangular window with an infinite duration signal.

\[ s_T(t) = w(t)s(t) \]

◆ For single spectral realizations, a multiply in time domain implies
  ❖ Fourier transforms convolve
  \[ S_T(f) = W(f) * S(f) \]
  ❖ and spectrum, \( S_{sT}(f) = |S_T(f)|^2 \)

◆ For ensemble average spectra, a multiply in time domain implies
  ❖ Power spectra convolve \( S_{sT}(f) = S_w(f) * S_s(f) \)
    ⇒ See Goodman's Gaussian moment theorem
    \[ S_{sn}(f) = \frac{No}{2} \left( 1 + \frac{S_{sinc}(f) * S_s(f)}{\int S_{sinc}(f)df} \right) \]

  ⇒ Signal Spectrum convolved with a unit area sinc\(^2\).
  ⇒ Unit area sinc\(^2\) normalization ensures noise PSD remains constant with dwell time variations
  ⇒ For a Gaussian signal spectrum (see backup charts)

\[ S_s(f) = \sqrt{2\text{CNR}_n} \exp \left( \frac{(f - fc)^2}{2\delta f^2} \right) \]
Monte Carlo Simulation Experimental Results for Arbitrary Integration Times Agree with Theory

- 10,000 Spectral Avg Monte Carlo Simulation
- Results agree with theory for arbitrary integration time

\[ \frac{T}{\tau_s} = 0.1 \]

\[ \frac{T}{\tau_s} = 1.0 \]

\[ \frac{T}{\tau_s} = 10 \]

\[ \text{Spectrum vs. Model} \]

\[ \text{Signal Periodogram} \]

\[ \text{Time Domain} \]

\[ \text{Heterodyne Signal} \]
Signal Detectivity

- Detectivity relates to sensor range performance

![Rad Vel Scatter Plot](image)

1000 pulse average
Data from 50 consecutive LOS wind estimates

Good estimates (CRLB) Anomalies

- Detectivity is defined as the ratio of the peak signal spectral height above the noise, to the rms fluctuations in the noise. Assume unit normalized NSD.
  - Spectral peak above noise is \( k \text{CNR}_n \)
    - where \( k \) is a constant close to 1
  - Noise rms is \( 1/\sqrt{N} \)
    - gamma distributed

- Consequently, the detectivity or Figure of Merit is given by

\[
FOM = k\sqrt{N\text{CNR}_n}
\]

- This FOM can be utilized to characterize the anomaly probability for a peak-detecting estimator algorithm

![Anomaly Probability](image)

Monte Carlo Simulation Results

- So for \( 2 < FOM < 3 \), \( PrA \sim 50\% \) depending on number of noise bins
Summary

- **Matched Filter CNR**
  \[ CNR_n = \frac{m_r}{M} \]
  Coherent photons build up CNRn
  Incoherent photons build up diversity

- **Diversity**
  - Fully coherent CW signal
    \[ M = 1 \]
  - Partially coherent CW signal
    \[ M^{-1} = \frac{1}{T} \int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau \]

- **Speckle Coherence Time**
  \[ \tau_s = \int |\gamma(\tau)|^2 d\tau \]

- **General Spectrum Model**
  \[ S_{\text{sn}}(f) = (No / 2) \left( 1 + \frac{S_{\text{sinc}}(f) * S_S(f)}{\int S_{\text{sinc}}(f) df} \right) \]

- **Peak above the normalized noise floor**
  - \( Sp = 1.0 \text{ CNR}_n \) for fully coherent signals
  - \( Sp \approx 1.2 \text{ CNR}_n \) for \( T/t_s = 1.0 \)
  - \( Sp = 1.414 \text{ CNR}_n \) for infinite duration incoherent signals

- **Signal Periodogram**

- **Signal Detectivity**
  \[ FOM = Sp \sqrt{N} \sim \text{CNR}_n \sqrt{N} \]

- **Coherent Receiver Sensitivity / Pulse**
  \[ \sim (1 \text{ coherent photon/} \eta_r) / \sqrt{N} \]